

ERLANGEN PROGRAMME AT LARGE: AN OVERVIEW

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Dedicated to Prof. Hans G. Feichtinger on the occasion of his 60th birthday

ABSTRACT. This is an overview of *Erlangen Programme at Large*. Study of objects and properties, which are invariant under a group action, is very fruitful far beyond the traditional geometry. In this paper we demonstrate this on the example of the group $SL_2(\mathbb{R})$. Starting from the conformal geometry we develop analytic functions and apply these to functional calculus. Finally we link this to quantum mechanics and conclude by a list of open problems.

CONTENTS

1. Introduction	2
1.1. Make a Guess in Three Attempts	3
1.2. Erlangen Programme at Large	4
2. Geometry	5
2.1. Hypercomplex Numbers	5
2.2. Cycles as Invariant Families	5
2.3. Invariants: Algebraic and Geometric	8
2.4. Joint Invariants: Orthogonality	9
2.5. Higher Order Joint Invariants: f-Orthogonality	10
2.6. Distance, Length and Perpendicularity	11
3. Linear Representations	13
3.1. Hypercomplex Characters	13
3.2. Induced Representations	14
3.3. Similarity and Correspondence: Ladder Operators	16
4. Covariant Transform	19
4.1. Extending Wavelet Transform	19
4.2. Examples of Covariant Transform	20
4.3. Symbolic Calculi	23
4.4. Inverse Covariant Transform	24
5. Analytic Functions	25
5.1. Induced Covariant Transform	25
5.2. Induced Wavelet Transform and Cauchy Integral	27
5.3. The Cauchy-Riemann (Dirac) and Laplace Operators	29
5.4. The Taylor Expansion	31
5.5. Wavelet Transform in the Unit Disk and Other Domains	31
6. Covariant and Contravariant Calculi	33
6.1. Intertwining Group Actions on Functions and Operators	34

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On leave from the Odessa University.

6.2. Jet Bundles and Prolongations of ρ_1	35
6.3. Spectrum and Spectral Mapping Theorem	36
6.4. Functional Model and Spectral Distance	38
6.5. Covariant Pencils of Operators	39
7. Quantum Mechanics	42
7.1. The Heisenberg Group and Its Automorphisms	44
7.2. p-Mechanic Formalism	46
7.3. Elliptic characters and Quantum Dynamics	48
7.4. Ladder Operators and Harmonic Oscillator	52
7.5. Hyperbolic Quantum Mechanics	54
7.6. Parabolic (Classical) Representations on the Phase Space	58
8. Open Problems	63
8.1. Geometry	63
8.2. Analytic Functions	63
8.3. Functional Calculus	63
8.4. Quantum Mechanics	64
Acknowledgement	65
References	65
Index	71

A mathematical idea should not be petrified in a formalised axiomatic settings, but should be considered instead as flowing as a river.

Sylvester (1878)

1. INTRODUCTION

The simplest objects with non-commutative (but still associative) multiplication may be 2×2 matrices with real entries. The subset of matrices of *determinant one* has the following properties:

- form a closed set under multiplication since $\det(AB) = \det A \cdot \det B$;
- the identity matrix is the set; and
- any such matrix has an inverse (since $\det A \neq 0$).

In other words those matrices form a *group*, the $SL_2(\mathbb{R})$ group [96]—one of the two most important Lie groups in analysis. The other group is the Heisenberg group [42]. By contrast the $ax + b$ group, which is often used to build wavelets, is only a subgroup of $SL_2(\mathbb{R})$, see the numerator in (1.1).

The simplest non-linear transforms of the real line—*linear-fractional* or *Möbius maps*—may also be associated with 2×2 matrices [8, Ch. 13]:

$$(1.1) \quad g : x \mapsto g \cdot x = \frac{ax + b}{cx + d}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad x \in \mathbb{R}.$$

An enjoyable calculation shows that the composition of two transforms (1.1) with different matrices g_1 and g_2 is again a Möbius transform with matrix the product $g_1 g_2$. In other words (1.1) it is a (left) action of $SL_2(\mathbb{R})$.

According to F. Klein's *Erlangen programme* (which was influenced by S. Lie) any geometry is dealing with invariant properties under a certain transitive group action. For example, we may ask: *What kinds of geometry are related to the $SL_2(\mathbb{R})$ action (1.1)?*

1.1. Make a Guess in Three Attempts. It is easy to see that the $\mathrm{SL}_2(\mathbb{R})$ action (1.1) makes sense also as a map of complex numbers $z = x + iy$, $i^2 = -1$ assuming the denominator is non-zero. Moreover, if $y > 0$ then $g \cdot z$ has a positive imaginary part as well, i.e. (1.1) defines a map from the upper half-plane to itself. Those transformations are isometries of the Lobachevsky half-plane.

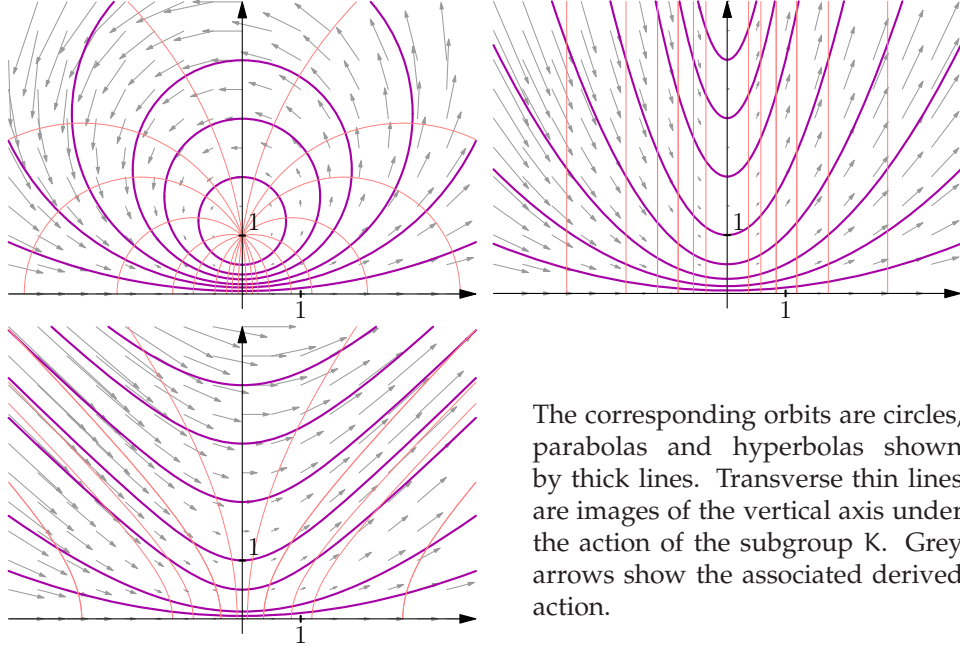
Three possible values -1 , 0 and 1 of $\sigma := \epsilon^2$ will be referred to here as *elliptic*, *parabolic* and *hyperbolic* cases respectively. We repeatedly meet such a division of various mathematical objects into three classes. They are named by the historically first example—the classification of conic sections—however the pattern persistently reproduces itself in many different areas: equations, quadratic forms, metrics, manifolds, operators, etc. We will abbreviate this separation as *EPH-classification*. The common origin of this fundamental division of any family with one-parameter can be seen from the simple picture of a coordinate line split by zero into negative and positive half-axes:



To understand the action (1.1) in all EPH cases we use the Iwasawa decomposition [96, § III.1] of $\mathrm{SL}_2(\mathbb{R}) = \mathrm{ANK}$ into three one-dimensional subgroups A , N and K :

(1.3)

By contrast, the action of the third matrix from the subgroup K sharply depends on σ , see Fig. 1. In elliptic, parabolic and hyperbolic cases K -orbits are circles, parabolas and (equilateral) hyperbolas correspondingly. Thin traversal lines in Fig. 1 join points of orbits for the same values of ϕ and grey arrows represent “local velocities”—vector fields of derived representations. We will describe some highlights of this geometry in Section 2.

FIGURE 1. Action of the subgroup K .

1.2. Erlangen Programme at Large. As we already mentioned the division of mathematics into areas is only apparent. Therefore it is unnatural to limit Erlangen programme only to “geometry”. We may continue to look for $SL_2(\mathbb{R})$ invariant objects in other related fields. For example, transform (1.1) generates unitary representations on certain L_2 spaces, cf. (1.1) and Section 3:

$$(1.4) \quad g : f(x) \mapsto \frac{1}{(cx + d)^m} f\left(\frac{ax + b}{cx + d}\right).$$

For $m = 1, 2, \dots$ the invariant subspaces of L_2 are Hardy and (weighted) Bergman spaces of complex analytic functions. All main objects of *complex analysis* (Cauchy and Bergman integrals, Cauchy-Riemann and Laplace equations, Taylor series etc.) may be obtaining in terms of invariants of the *discrete series* representations of $SL_2(\mathbb{R})$ [69, § 3]. Moreover two other series (*principal* and *complementary* [96]) play the similar rôles for hyperbolic and parabolic cases [69, 82]. This will be discussed in Sections 4 and 5.

Moving further we may observe that transform (1.1) is defined also for an element x in any algebra \mathfrak{A} with a unit 1 as soon as $(cx + d1) \in \mathfrak{A}$ has an inverse. If \mathfrak{A} is equipped with a topology, e.g. is a Banach algebra, then we may study a *functional calculus* for element x [75] in this way. It is defined as an *intertwining operator* between the representation (1.4) in a space of analytic functions and a similar representation in a left \mathfrak{A} -module. We will consider the Section 6.

In the spirit of Erlangen programme such functional calculus is still a geometry, since it is dealing with invariant properties under a group action. However even for a simplest non-normal operator, e.g. a Jordan block of the length k , the obtained space is not like a space of point but is rather a space of k -th *jets* [75]. Such non-point behaviour is often attributed to *non-commutative geometry* and Erlangen programme provides an important input on this fashionable topic [69].

It is noteworthy that ideas of F. Klein and S. Lie are spread more in physics than in mathematics: it is a common viewpoint that laws of nature shall be invariant

under certain transformations. Yet systematic use of Erlangen approach can bring new results even in this domain as we demonstrate in Section 7. There are still many directions to extend the present work thus we will conclude by a list of some open problems in Section 8.

Of course, there is no reasons to limit Erlangen programme to $SL_2(\mathbb{R})$ group only, other groups may be more suitable in different situations. However $SL_2(\mathbb{R})$ still possesses a big unexplored potential and is a good object to start with.

2. GEOMETRY

We start from the natural domain of the Erlangen Programme—geometry. Systematic use of this ideology allows to obtain new results even for very classical objects like circles.

2.1. Hypercomplex Numbers. Firstly we wish to demonstrate that hypercomplex numbers appear very naturally from a study of $SL_2(\mathbb{R})$ action on the homogeneous spaces [85]. We begin from the standard definitions.

Let H be a subgroup of a group G . Let $X = G/H$ be the corresponding homogeneous space and $s : X \rightarrow G$ be a smooth section [55, § 13.2], which is a left inverse to the natural projection $p : G \rightarrow X$. The choice of s is inessential in the sense that by a smooth map $X \rightarrow X$ we can always reduce one to another. We define a map $r : G \rightarrow H$ associated to p and s from the identities:

$$(2.1) \quad r(g) = (s(x))^{-1}g, \quad \text{where } x = p(g) \in X.$$

Note that X is a left homogeneous space with the G -action defined in terms of p and s as follows:

$$(2.2) \quad g : x \mapsto g \cdot x = p(g * s(x)),$$

Example 2.1 ([85]). For $G = SL_2(\mathbb{R})$, as well as for other semisimple groups, it is common to consider only the case of H being the maximal compact subgroup K . However in this paper we admit H to be any one-dimensional subgroup. Then X is a two-dimensional manifold and for any choice of H we define [64, Ex. 3.7(a)]:

$$(2.3) \quad s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2, v > 0.$$

Any continuous one-dimensional subgroup $H \in SL_2(\mathbb{R})$ is conjugated to one of the following:

$$(2.4) \quad K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, t \in (-\pi, \pi] \right\},$$

$$(2.5) \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\},$$

$$(2.6) \quad A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Then [85] the action (2.2) of $SL_2(\mathbb{R})$ on $X = SL_2(\mathbb{R})/H$ coincides with Möbius transformations (1.1) on complex, dual and double numbers respectively.

2.2. Cycles as Invariant Families. We wish to consider all three hypercomplex systems at the same time, the following definition is very helpful for this.

Definition 2.2. The common name *cycle* [125] is used to denote circles, parabolas and hyperbolas (as well as straight lines as their limits) in the respective EPH case.

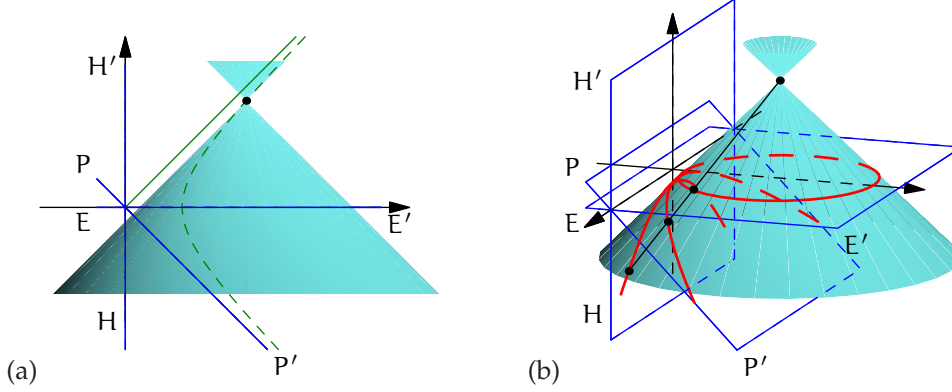


FIGURE 2. K-orbits as conic sections: circles are sections by the plane EE' ; parabolas are sections by PP' ; hyperbolas are sections by HH' . Points on the same generator of the cone correspond to the same value of ϕ .

It is well known that any cycle is a *conic sections* and an interesting observation is that corresponding K-orbits are in fact sections of the same two-sided right-angle cone, see Fig. 2. Moreover, each straight line generating the cone, see Fig. 2(b), is crossing corresponding EPH K-orbits at points with the same value of parameter ϕ from (1.3). In other words, all three types of orbits are generated by the rotations of this generator along the cone.

K-orbits are K-invariant in a trivial way. Moreover since actions of both A and N for any σ are extremely “shape-preserving” we find natural invariant objects of the Möbius map:

Theorem 2.3. *The family of all cycles from Definition 2.2 is invariant under the action (1.1).*

According to Erlangen ideology we should now study invariant properties of cycles.

Fig. 2 suggests that we may get a unified treatment of cycles in all EPH cases by consideration of a higher dimension spaces. The standard mathematical method is to declare objects under investigations (cycles in our case, functions in functional analysis, etc.) to be simply points of some bigger space. This space should be equipped with an appropriate structure to hold externally information which were previously inner properties of our objects.

A generic cycle is the set of points $(u, v) \in \mathbb{R}^2$ defined for all values of σ by the equation

$$(2.7) \quad k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0.$$

This equation (and the corresponding cycle) is defined by a point (k, l, n, m) from a *projective space* \mathbb{P}^3 , since for a scaling factor $\lambda \neq 0$ the point $(\lambda k, \lambda l, \lambda n, \lambda m)$ defines an equation equivalent to (2.7). We call \mathbb{P}^3 the *cycle space* and refer to the initial \mathbb{R}^2 as the *point space*.

In order to get a connection with Möbius action (1.1) we arrange numbers (k, l, n, m) into the matrix

$$(2.8) \quad C_{\sigma}^s = \begin{pmatrix} l + \mathfrak{I}sn & -m \\ k & -l + \mathfrak{I}sn \end{pmatrix},$$

with a new hypercomplex unit \mathfrak{I} and an additional parameter s usually equal to ± 1 . The values of $\mathfrak{I} := \mathfrak{I}^2$ is $-1, 0$ or 1 independently from the value of σ . The

matrix (2.8) is the cornerstone of an extended *Fillmore–Springer–Cnops construction* (FSCc) [19].

The significance of FSCc in Erlangen framework is provided by the following result:

Theorem 2.4. *The image \tilde{C}_α^s of a cycle C_α^s under transformation (1.1) with $g \in \text{SL}_2(\mathbb{R})$ is given by similarity of the matrix (2.8):*

$$(2.9) \quad \tilde{C}_\alpha^s = g C_\alpha^s g^{-1}.$$

In other words FSCc (2.8) intertwines Möbius action (1.1) on cycles with linear map (2.9).

There are several ways to prove (2.9): either by a brute force calculation (fortunately performed by a CAS) [82] or through the related orthogonality of cycles [19], see the end of the next Subsection 2.3.

The important observation here is that our extended version of FSCc (2.8) uses an hypercomplex unit \mathfrak{i} , which is not related to \mathfrak{i} defining the appearance of cycles on plane. In other words any EPH type of geometry in the cycle space \mathbb{P}^3 admits drawing of cycles in the point space \mathbb{R}^2 as circles, parabolas or hyperbolas. We may think on points of \mathbb{P}^3 as ideal cycles while their depictions on \mathbb{R}^2 are only their shadows on the wall of Plato's cave.

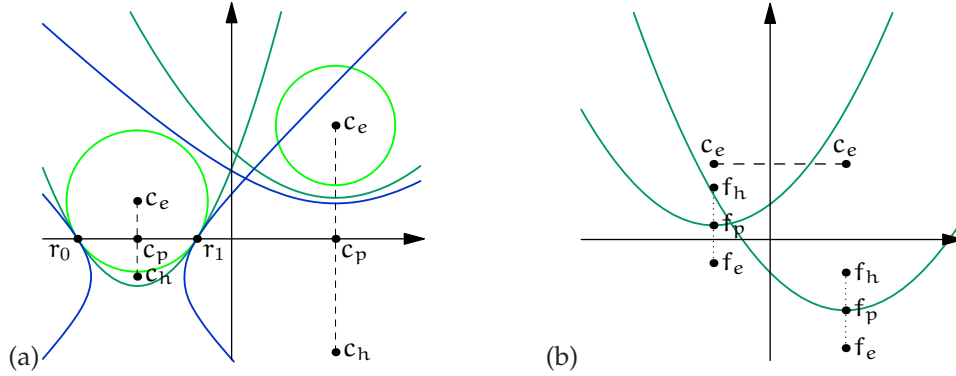


FIGURE 3. (a) Different EPH implementations of the same cycles defined by quadruples of numbers.
(b) Centres and foci of two parabolas with the same focal length.

Fig. 3(a) shows the same cycles drawn in different EPH styles. Points $c_{e,p,h} = (\frac{1}{k}, -\tilde{\sigma}\frac{n}{k})$ are their respective e/p/h-centres. They are related to each other through several identities:

$$(2.10) \quad c_e = \bar{c}_h, \quad c_p = \frac{1}{2}(c_e + c_h).$$

Fig. 3(b) presents two cycles drawn as parabolas, they have the same focal length $\frac{n}{2k}$ and thus their e-centres are on the same level. In other words *concentric* parabolas are obtained by a vertical shift, not scaling as an analogy with circles or hyperbolas may suggest.

Fig. 3(b) also presents points, called e/p/h-foci:

$$(2.11) \quad f_{e,p,h} = \left(\frac{1}{k}, -\frac{\det C_\alpha^s}{2nk} \right),$$

which are independent of the sign of s . If a cycle is depicted as a parabola then h-focus, p-focus, e-focus are correspondingly geometrical focus of the parabola, its vertex, and the point on the directrix nearest to the vertex.

As we will see, cf. Theorems 2.6 and 2.8, all three centres and three foci are useful attributes of a cycle even if it is drawn as a circle.

2.3. Invariants: Algebraic and Geometric. We use known algebraic invariants of matrices to build appropriate geometric invariants of cycles. It is yet another demonstration that any division of mathematics into subjects is only illusive.

For 2×2 matrices (and thus cycles) there are only two essentially different invariants under similarity (2.9) (and thus under Möbius action (1.1)): the *trace* and the *determinant*. The latter was already used in (2.11) to define cycle's foci. However due to projective nature of the cycle space \mathbb{P}^3 the absolute values of trace or determinant are irrelevant, unless they are zero.

Alternatively we may have a special arrangement for normalisation of quadruples (k, l, n, m) . For example, if $k \neq 0$ we may normalise the quadruple to $(1, \frac{1}{k}, \frac{n}{k}, \frac{m}{k})$ with highlighted cycle's centre. Moreover in this case $\det C_{\check{\sigma}}^s$ is equal to the square of cycle's radius, cf. Section 2.6. Another normalisation $\det C_{\check{\sigma}}^s = 1$ is used in [58] to get a nice condition for touching circles. Moreover, the Kirillov normalisation is preserved by the conjugation (2.9).

We still get important characterisation even with non-normalised cycles, e.g., invariant classes (for different $\check{\sigma}$) of cycles are defined by the condition $\det C_{\check{\sigma}}^s = 0$. Such a class is parametrised only by two real numbers and as such is easily attached to certain point of \mathbb{R}^2 . For example, the cycle $C_{\check{\sigma}}^s$ with $\det C_{\check{\sigma}}^s = 0$, $\check{\sigma} = -1$ drawn elliptically represent just a point $(\frac{1}{k}, \frac{n}{k})$, i.e. (elliptic) zero-radius circle. The same condition with $\check{\sigma} = 1$ in hyperbolic drawing produces a null-cone originated at point $(\frac{1}{k}, \frac{n}{k})$:

$$(u - \frac{1}{k})^2 - (v - \frac{n}{k})^2 = 0,$$

i.e. a zero-radius cycle in hyperbolic metric.

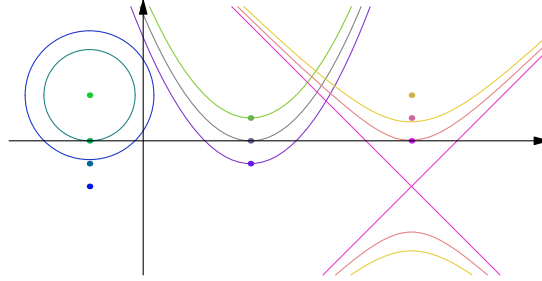


FIGURE 4. Different σ -implementations of the same $\check{\sigma}$ -zero-radius cycles and corresponding foci.

In general for every notion there are (at least) nine possibilities: three EPH cases in the cycle space times three EPH realisations in the point space. Such nine cases for “zero radius” cycles is shown in Fig. 4. For example, p-zero-radius cycles in any implementation touch the real axis.

This “touching” property is a manifestation of the *boundary effect* in the upper-half plane geometry. The famous question on hearing drum’s shape has a sister:

Can we see/feel the boundary from inside a domain?

Both orthogonality relations described below are “boundary aware” as well. It is not surprising after all since $SL_2(\mathbb{R})$ action on the upper-half plane was obtained as an extension of its action (1.1) on the boundary.

According to the categorical viewpoint internal properties of objects are of minor importance in comparison to their relations with other objects from the same class. As an illustration we may put the proof of Theorem 2.4 sketched at the

end of the next section. Thus from now on we will look for invariant relations between two or more cycles.

2.4. Joint Invariants: Orthogonality. The most expected relation between cycles is based on the following Möbius invariant “inner product” build from a trace of product of two cycles as matrices:

$$(2.12) \quad \langle C_{\check{\sigma}}^s, \tilde{C}_{\check{\sigma}}^s \rangle = \text{tr}(C_{\check{\sigma}}^s \tilde{C}_{\check{\sigma}}^s)$$

By the way, an inner product of this type is used, for example, in GNS construction to make a Hilbert space out of C^* -algebra. The next standard move is given by the following definition.

Definition 2.5. Two cycles are called $\check{\sigma}$ -orthogonal if $\langle C_{\check{\sigma}}^s, \tilde{C}_{\check{\sigma}}^s \rangle = 0$.

The orthogonality relation is preserved under the Möbius transformations, thus this is an example of a *joint invariant* of two cycles. For the case of $\check{\sigma} = 1$, i.e. when geometries of the cycle and point spaces are both either elliptic or hyperbolic, such an orthogonality is the standard one, defined in terms of angles between tangent lines in the intersection points of two cycles. However in the remaining seven ($= 9 - 2$) cases the innocent-looking Definition 2.5 brings unexpected relations.

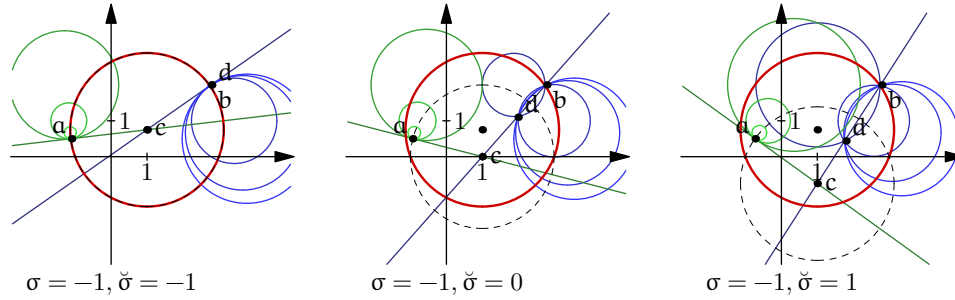


FIGURE 5. Orthogonality of the first kind in the elliptic point space.

Each picture presents two groups (green and blue) of cycles which are orthogonal to the red cycle $C_{\check{\sigma}}^s$. Point b belongs to $C_{\check{\sigma}}^s$ and the family of blue cycles passing through b is orthogonal to $C_{\check{\sigma}}^s$. They all also intersect in the point d which is the inverse of b in $C_{\check{\sigma}}^s$. Any orthogonality is reduced to the usual orthogonality with a new (“ghost”) cycle (shown by the dashed line), which may or may not coincide with $C_{\check{\sigma}}^s$. For any point a on the “ghost” cycle the orthogonality is reduced to the local notion in the terms of tangent lines at the intersection point. Consequently such a point a is always the inverse of itself.

Elliptic (in the point space) realisations of Definition 2.5, i.e. $\sigma = -1$ is shown in Fig. 5. The left picture corresponds to the elliptic cycle space, e.g. $\check{\sigma} = -1$. The orthogonality between the red circle and any circle from the blue or green families is given in the usual Euclidean sense. The central (parabolic in the cycle space) and the right (hyperbolic) pictures show non-local nature of the orthogonality. There are analogues pictures in parabolic and hyperbolic point spaces as well, see [82, 90].

This orthogonality may still be expressed in the traditional sense if we will associate to the red circle the corresponding “ghost” circle, which shown by the dashed

line in Fig. 5. To describe ghost cycle we need the *Heaviside function* $\chi(\sigma)$:

$$(2.13) \quad \chi(t) = \begin{cases} 1, & t \geq 0; \\ -1, & t < 0. \end{cases}$$

Theorem 2.6. *A cycle is $\check{\sigma}$ -orthogonal to cycle $C_{\check{\sigma}}^s$ if it is orthogonal in the usual sense to the σ -realisation of “ghost” cycle $\hat{C}_{\check{\sigma}}^s$, which is defined by the following two conditions:*

- (i) $\chi(\sigma)$ -centre of $\hat{C}_{\check{\sigma}}^s$ coincides with $\check{\sigma}$ -centre of $C_{\check{\sigma}}^s$.
- (ii) Cycles $\hat{C}_{\check{\sigma}}^s$ and $C_{\check{\sigma}}^s$ have the same roots, moreover $\det \hat{C}_{\check{\sigma}}^1 = \det C_{\check{\sigma}}^{X(\check{\sigma})}$.

The above connection between various centres of cycles illustrates their meaningfulness within our approach.

One can easily check the following orthogonality properties of the zero-radius cycles defined in the previous section:

- (i) Due to the identity $\langle C_{\check{\sigma}}^s, C_{\check{\sigma}}^s \rangle = \det C_{\check{\sigma}}^s$ zero-radius cycles are self-orthogonal (isotropic) ones.
- (ii) A cycle $C_{\check{\sigma}}^s$ is σ -orthogonal to a zero-radius cycle $Z_{\check{\sigma}}^s$ if and only if $C_{\check{\sigma}}^s$ passes through the σ -centre of $Z_{\check{\sigma}}^s$.

As we will see, in parabolic case there is a more suitable notion of an infinitesimal cycle which can be used instead of zero-radius ones.

2.5. Higher Order Joint Invariants: f-Orthogonality. With appetite already wet one may wish to build more joint invariants. Indeed for any polynomial $p(x_1, x_2, \dots, x_n)$ of several non-commuting variables one may define an invariant joint disposition of n cycles ${}^j C_{\check{\sigma}}^s$ by the condition:

$$\text{tr } p({}^1 C_{\check{\sigma}}^s, {}^2 C_{\check{\sigma}}^s, \dots, {}^n C_{\check{\sigma}}^s) = 0.$$

However it is preferable to keep some geometrical meaning of constructed notions.

An interesting observation is that in the matrix similarity of cycles (2.9) one may replace element $g \in \text{SL}_2(\mathbb{R})$ by an arbitrary matrix corresponding to another cycle. More precisely the product $C_{\check{\sigma}}^s \tilde{C}_{\check{\sigma}}^s C_{\check{\sigma}}^s$ is again the matrix of the form (2.8) and thus may be associated to a cycle. This cycle may be considered as the reflection of $\tilde{C}_{\check{\sigma}}^s$ in $C_{\check{\sigma}}^s$.

Definition 2.7. A cycle $C_{\check{\sigma}}^s$ is *f-orthogonal* (focal orthogonal) to a cycle $\tilde{C}_{\check{\sigma}}^s$ if the reflection of $\tilde{C}_{\check{\sigma}}^s$ in $C_{\check{\sigma}}^s$ is orthogonal (in the sense of Definition 2.5) to the real line. Analytically this is defined by:

$$(2.14) \quad \text{tr}(C_{\check{\sigma}}^s \tilde{C}_{\check{\sigma}}^s C_{\check{\sigma}}^s R_{\check{\sigma}}^s) = 0.$$

Due to invariance of all components in the above definition f-orthogonality is a Möbius invariant condition. Clearly this is not a symmetric relation: if $C_{\check{\sigma}}^s$ is f-orthogonal to $\tilde{C}_{\check{\sigma}}^s$ then $\tilde{C}_{\check{\sigma}}^s$ is not necessarily f-orthogonal to $C_{\check{\sigma}}^s$.

Fig. 6 illustrates f-orthogonality in the elliptic point space. By contrast with Fig. 5 it is not a local notion at the intersection points of cycles for all $\check{\sigma}$. However it may be again clarified in terms of the appropriate s-ghost cycle, cf. Theorem 2.6.

Theorem 2.8. *A cycle is f-orthogonal to a cycle $C_{\check{\sigma}}^s$ if its orthogonal in the traditional sense to its f-ghost cycle $\tilde{C}_{\check{\sigma}}^{\check{\sigma}} = C_{\check{\sigma}}^{X(\sigma)} \mathbb{R}_{\check{\sigma}}^{\check{\sigma}} C_{\check{\sigma}}^{X(\sigma)}$, which is the reflection of the real line in $C_{\check{\sigma}}^{X(\sigma)}$ and χ is the Heaviside function (2.13). Moreover*

- (i) $\chi(\sigma)$ -Centre of $\tilde{C}_{\check{\sigma}}^{\check{\sigma}}$ coincides with the $\check{\sigma}$ -focus of $C_{\check{\sigma}}^s$, consequently all lines f-orthogonal to $C_{\check{\sigma}}^s$ are passing the respective focus.
- (ii) Cycles $C_{\check{\sigma}}^s$ and $\tilde{C}_{\check{\sigma}}^{\check{\sigma}}$ have the same roots.

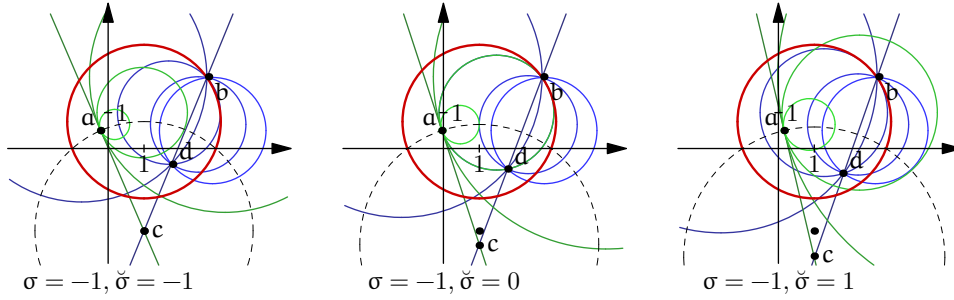


FIGURE 6. Focal orthogonality for circles. To highlight both similarities and distinctions with the ordinary orthogonality we use the same notations as that in Fig. 5.

Note the above intriguing interplay between cycle's centres and foci. Although f-orthogonality may look exotic it will naturally appear in the end of next Section again.

Of course, it is possible to define another interesting higher order joint invariants of two or even more cycles.

2.6. Distance, Length and Perpendicularity. *Geometry* in the plain meaning of this word deals with *distances* and *lengths*. Can we obtain them from cycles?

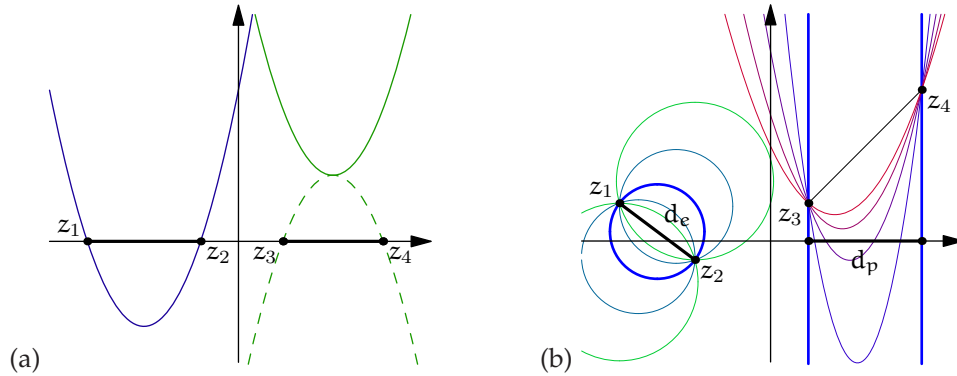


FIGURE 7. (a) The square of the parabolic diameter is the square of the distance between roots if they are real (z_1 and z_2), otherwise the negative square of the distance between the adjoint roots (z_3 and z_4).

(b) Distance as extremum of diameters in elliptic (z_1 and z_2) and parabolic (z_3 and z_4) cases.

We mentioned already that for circles normalised by the condition $k = 1$ the value $\det C_{\check{\sigma}}^s = \langle C_{\check{\sigma}}^s, C_{\check{\sigma}}^s \rangle$ produces the square of the traditional circle radius. Thus we may keep it as the definition of the $\check{\sigma}$ -radius for any cycle. But then we need to accept that in the parabolic case the radius is the (Euclidean) distance between (real) roots of the parabola, see Fig. 7(a).

Having radii of circles already defined we may use them for other measurements in several different ways. For example, the following variational definition may be used:

Definition 2.9. The *distance* between two points is the extremum of diameters of all cycles passing through both points, see Fig. 7(b).

If $\sigma = \sigma$ this definition gives in all EPH cases the following expression for a distance $d_{e,p,h}(u, v)$ between endpoints of any vector $w = u + iv$:

$$(2.15) \quad d_{e,p,h}(u, v)^2 = (u + iv)(u - iv) = u^2 - \sigma v^2.$$

The parabolic distance $d_p^2 = u^2$, see Fig. 7(b), algebraically sits between d_e and d_h according to the general principle (1.2) and is widely accepted [125]. However one may be unsatisfied by its degeneracy.

An alternative measurement is motivated by the fact that a circle is the set of equidistant points from its centre. However the choice of “centre” is now rich: it may be either point from three centres (2.10) or three foci (2.11).

Definition 2.10. The *length* of a directed interval \overrightarrow{AB} is the radius of the cycle with its *centre* (denoted by $l_c(\overrightarrow{AB})$) or *focus* (denoted by $l_f(\overrightarrow{AB})$) at the point A which passes through B.

This definition is less common and have some unusual properties like non-symmetry: $l_f(\overrightarrow{AB}) \neq l_f(\overrightarrow{BA})$. However it comfortably fits the Erlangen programme due to its $SL_2(\mathbb{R})$ -conformal invariance:

Theorem 2.11 ([82]). Let l denote either the EPH distances (2.15) or any length from Definition 2.10. Then for fixed $y, y' \in \mathbb{R}^\sigma$ the limit:

$$\lim_{t \rightarrow 0} \frac{l(g \cdot y, g \cdot (y + ty'))}{l(y, y + ty')}, \quad \text{where } g \in SL_2(\mathbb{R}),$$

exists and its value depends only from y and g and is independent from y' .

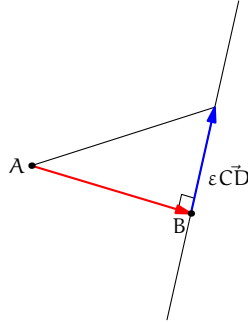


FIGURE 8. Perpendicular as the shortest route to a line.

We may return from distances to angles recalling that in the Euclidean space a perpendicular provides the shortest root from a point to a line, see Fig. 8.

Definition 2.12. Let l be a length or distance. We say that a vector \overrightarrow{AB} is l -perpendicular to a vector \overrightarrow{CD} if function $l(\overrightarrow{AB} + \varepsilon \overrightarrow{CD})$ of a variable ε has a local extremum at $\varepsilon = 0$.

A pleasant surprise is that l_f -perpendicularity obtained through the length from focus (Definition 2.10) coincides with already defined in Section 2.5 f -orthogonality as follows from Thm. 2.8(i). It is also possible [59] to make $SL_2(\mathbb{R})$ action isometric in all three cases.

Further details of the refreshing geometry of Möbius transformation can be found in the paper [82] and the book [90].

All these study are waiting to be generalised to high dimensions, quaternions and Clifford algebras provide a suitable language for this [82, 108].

3. LINEAR REPRESENTATIONS

A consideration of the symmetries in analysis is natural to start from *linear representations*. The previous geometrical actions (1.1) can be naturally extended to such representations by induction [55, § 13.2; 64, § 3.1] from a representation of a subgroup H . If H is one-dimensional then its irreducible representation is a character, which is always supposed to be a complex valued. However hypercomplex number naturally appeared in the $SL_2(\mathbb{R})$ action (1.1), see Subsection 2.1 and [85], why shall we admit only $i^2 = -1$ to deliver a character then?

3.1. Hypercomplex Characters. As we already mentioned the typical discussion of induced representations of $SL_2(\mathbb{R})$ is centred around the case $H = K$ and a complex valued character of K . A linear transformation defined by a matrix (2.4) in K is a rotation of \mathbb{R}^2 by the angle t . After identification $\mathbb{R}^2 = \mathbb{C}$ this action is given by the multiplication e^{it} , with $i^2 = -1$. The rotation preserve the (elliptic) metric given by:

$$(3.1) \quad x^2 + y^2 = (x + iy)(x - iy).$$

Therefore the orbits of rotations are circles, any line passing the origin (a “spoke”) is rotated by the angle t , see Fig. 9.

Dual and double numbers produces the most straightforward adaptation of this result.

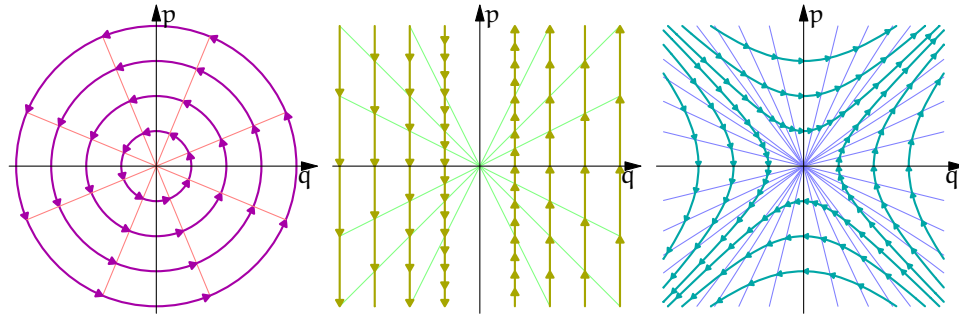


FIGURE 9. Rotations of algebraic wheels, i.e. the multiplication by e^{it} : elliptic (E), trivial parabolic (P_0) and hyperbolic (H). All blue orbits are defined by the identity $x^2 - \iota^2 y^2 = r^2$. Thin “spokes” (straight lines from the origin to a point on the orbit) are “rotated” from the real axis. This is symplectic linear transformations of the classical phase space as well.

Proposition 3.1. *The following table show correspondences between three types of algebraic characters:*

<i>Elliptic</i>	<i>Parabolic</i>	<i>Hyperbolic</i>
$i^2 = -1$	$\varepsilon^2 = 0$	$j^2 = 1$
$w = x + iy$	$w = x + \varepsilon y$	$w = x + jy$
$\bar{w} = x - iy$	$\bar{w} = x - \varepsilon y$	$\bar{w} = x - jy$
$e^{it} = \cos t + i \sin t$	$e^{\varepsilon t} = 1 + \varepsilon t$	$e^{jt} = \cosh t + j \sinh t$
$ w _e^2 = w\bar{w} = x^2 + y^2$	$ w _p^2 = w\bar{w} = x^2$	$ w _h^2 = w\bar{w} = x^2 - y^2$
$\arg w = \tan^{-1} \frac{y}{x}$	$\arg w = \frac{y}{x}$	$\arg w = \tanh^{-1} \frac{y}{x}$
<i>unit circle</i> $ w _e^2 = 1$	<i>“unit” strip</i> $x = \pm 1$	<i>unit hyperbola</i> $ w _h^2 = 1$

Geometrical action of multiplication by $e^{t\epsilon}$ is drawn in Fig. 9 for all three cases.

Explicitly parabolic rotations associated with $e^{\epsilon t}$ acts on dual numbers as follows:

$$(3.2) \quad e^{\epsilon x} : a + \epsilon b \mapsto a + \epsilon(ax + b).$$

This links the parabolic case with the Galilean group [125] of symmetries of the classic mechanics, with the absolute time disconnected from space.

The obvious algebraic similarity and the connection to classical kinematic is a wide spread justification for the following viewpoint on the parabolic case, cf. [40, 125]:

- the parabolic trigonometric functions are trivial:

$$(3.3) \quad \cosh t = \pm 1, \quad \sinh t = t;$$

- the parabolic distance is independent from y if $x \neq 0$:

$$(3.4) \quad x^2 = (x + \epsilon y)(x - \epsilon y);$$

- the polar decomposition of a dual number is defined by [125, App. C(30')]:

$$(3.5) \quad u + \epsilon v = u(1 + \epsilon \frac{v}{u}), \quad \text{thus} \quad |u + \epsilon v| = u, \quad \arg(u + \epsilon v) = \frac{v}{u};$$

- the parabolic wheel looks rectangular, see Fig. 9.

Those algebraic analogies are quite explicit and widely accepted as an ultimate source for parabolic trigonometry [40, 97, 125]. Moreover, those three rotations are all non-isomorphic symplectic linear transformations of the phase space, which makes them useful in the context of classical and quantum mechanics [87, 88], see Section 7. There exist also alternative characters [79] based on Möbius transformations with geometric motivation and connections to equations of mathematical physics.

3.2. Induced Representations. Let G be a group, H be its closed subgroup with the corresponding homogeneous space $X = G/H$ with an invariant measure. We are using notations and definitions of maps $p : G \rightarrow X$, $s : X \rightarrow G$ and $r : G \rightarrow H$ from Subsection 2.1. Let χ be an irreducible representation of H in a vector space V , then it induces a representation of G in the sense of Mackey [55, § 13.2]. This representation has the realisation ρ_χ in the space $L_2(X)$ of V -valued functions by the formula [55, § 13.2.(7)–(9)]:

$$(3.6) \quad [\rho_\chi(g)f](x) = \chi(r(g^{-1} * s(x)))f(g^{-1} \cdot x),$$

where $g \in G$, $x \in X$, $h \in H$ and $r : G \rightarrow H$, $s : X \rightarrow G$ are maps defined above; $*$ denotes multiplication on G and \cdot denotes the action (2.2) of G on X .

Consider this scheme for representations of $SL_2(\mathbb{R})$ induced from characters of its one-dimensional subgroups. We can notice that only the subgroup K requires a complex valued character due to the fact of its compactness. For subgroups N' and A' we can consider characters of all three types—elliptic, parabolic and hyperbolic. Therefore we have seven essentially different induced representations. We will write explicitly only three of them here.

Example 3.2. Consider the subgroup $H = K$, due to its compactness we are limited to complex valued characters of K only. All of them are of the form χ_k :

$$(3.7) \quad \chi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-ikt}, \quad \text{where } k \in \mathbb{Z}.$$

Using the explicit form (2.3) of the map s we find the map r given in (2.1) as follows:

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{c^2 + d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \in K.$$

Therefore:

$$r(g^{-1} * s(u, v)) = \frac{1}{\sqrt{(cu + d)^2 + (cv)^2}} \begin{pmatrix} cu + d & -cv \\ cv & cu + d \end{pmatrix}, \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Substituting this into (3.7) and combining with the Möbius transformation of the domain (1.1) we get the explicit realisation ρ_k of the induced representation (3.6):

$$(3.8) \quad \rho_k(g)f(w) = \frac{|cw + d|^k}{(cw + d)^k} f\left(\frac{aw + b}{cw + d}\right), \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = u + iv.$$

This representation acts on complex valued functions in the upper half-plane $\mathbb{R}_+^2 = \text{SL}_2(\mathbb{R})/K$ and belongs to the discrete series [96, § IX.2]. It is common to get rid of the factor $|cw + d|^k$ from that expression in order to keep analyticity and we will follow this practise for a convenience as well.

Example 3.3. In the case of the subgroup N there is a wider choice of possible characters.

- (i) Traditionally only complex valued characters of the subgroup N are considered, they are:

$$(3.9) \quad \chi_\tau^{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{i\tau t}, \quad \text{where } \tau \in \mathbb{R}.$$

A direct calculation shows that:

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} \in N'.$$

Thus:

$$(3.10) \quad r(g^{-1} * s(u, v)) = \begin{pmatrix} 1 & 0 \\ \frac{cv}{d+cu} & 1 \end{pmatrix}, \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A substitution of this value into the character (3.9) together with the Möbius transformation (1.1) we obtain the next realisation of (3.6):

$$\rho_\tau^{\mathbb{C}}(g)f(w) = \exp\left(i\frac{\tau cv}{cu + d}\right) f\left(\frac{aw + b}{cw + d}\right), \quad \text{where } w = u + \varepsilon v, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The representation acts on the space of *complex* valued functions on the upper half-plane \mathbb{R}_+^2 , which is a subset of *dual* numbers as a homogeneous space $\text{SL}_2(\mathbb{R})/N'$. The mixture of complex and dual numbers in the same expression is confusing.

- (ii) The parabolic character χ_τ with the algebraic flavour is provided by multiplication (3.2) with the dual number:

$$\chi_\tau \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{\varepsilon \tau t} = 1 + \varepsilon \tau t, \quad \text{where } \tau \in \mathbb{R}.$$

If we substitute the value (3.10) into this character, then we receive the representation:

$$\rho_\tau(g)f(w) = \left(1 + \varepsilon \frac{\tau cv}{cu + d}\right) f\left(\frac{aw + b}{cw + d}\right),$$

where w, τ and g are as above. The representation is defined on the space of dual numbers valued functions on the upper half-plane of dual numbers. Thus expression contains only dual numbers with their usual algebraic operations. Thus it is linear with respect to them.

All characters in the previous Example are unitary. Then the general scheme of induced representations [55, § 13.2] implies their unitarity in proper senses.

Theorem 3.4 ([85]). *Both representations of $SL_2(\mathbb{R})$ from Example 3.3 are unitary on the space of function on the upper half-plane \mathbb{R}_+^2 of dual numbers with the inner product:*

$$(3.11) \quad \langle f_1, f_2 \rangle = \int_{\mathbb{R}_+^2} f_1(w) \bar{f}_2(w) \frac{du dv}{v^2}, \quad \text{where } w = u + \varepsilon v,$$

and we use the conjugation and multiplication of functions' values in algebras of complex and dual numbers for representations $\rho_\tau^\mathbb{C}$ and ρ_τ respectively.

The inner product (3.11) is positive defined for the representation $\rho_\tau^\mathbb{C}$ but is not for the other. The respective spaces are parabolic cousins of the Krein spaces [5], which are hyperbolic in our sense.

3.3. Similarity and Correspondence: Ladder Operators. From the above observation we can deduce the following empirical principle, which has a heuristic value.

Principle 3.5 (Similarity and correspondence). (i) *Subgroups K, N' and A' play a similar rôle in the structure of the group $SL_2(\mathbb{R})$ and its representations.*
(ii) *The subgroups shall be swapped simultaneously with the respective replacement of hypercomplex unit ι .*

The first part of the Principle (similarity) does not look sound alone. It is enough to mention that the subgroup K is compact (and thus its spectrum is discrete) while two other subgroups are not. However in a conjunction with the second part (correspondence) the Principle have received the following confirmations so far, see [85] for details:

- The action of $SL_2(\mathbb{R})$ on the homogeneous space $SL_2(\mathbb{R})/H$ for $H = K, N'$ or A' is given by linear-fractional transformations of complex, dual or double numbers respectively.
- Subgroups K, N' or A' are isomorphic to the groups of unitary rotations of respective unit cycles in complex, dual or double numbers.
- Representations induced from subgroups K, N' or A' are unitary if the inner product spaces of functions with values in complex, dual or double numbers.

Remark 3.6. The principle of similarity and correspondence resembles *supersymmetry* between bosons and fermions in particle physics, but we have similarity between three different types of entities in our case.

Let us give another illustration to the Principle. Consider the Lie algebra \mathfrak{sl}_2 of the group $SL_2(\mathbb{R})$. Pick up the following basis in \mathfrak{sl}_2 [117, § 8.1]:

$$(3.12) \quad A = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The commutation relations between the elements are:

$$(3.13) \quad [Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2}Z.$$

Let ρ be a representation of the group $SL_2(\mathbb{R})$ in a space V . Consider the derived representation $d\rho$ of the Lie algebra \mathfrak{sl}_2 [96, § VI.1] and denote $\tilde{X} = d\rho(X)$ for

$X \in \mathfrak{sl}_2$. To see the structure of the representation ρ we can decompose the space V into eigenspaces of the operator \tilde{X} for some $X \in \mathfrak{sl}_2$, cf. the Taylor series in Section 5.4.

3.3.1. Elliptic Ladder Operators. It would not be surprising that we are going to consider three cases: Let $X = Z$ be a generator of the subgroup K (2.4). Since this is a compact subgroup the corresponding eigenspaces $\tilde{Z}v_k = ikv_k$ are parametrised by an integer $k \in \mathbb{Z}$. The raising/lowering or ladder operators L^\pm [96, § VI.2; 117, § 8.2] are defined by the following commutation relations:

$$(3.14) \quad [\tilde{Z}, L^\pm] = \lambda_\pm L^\pm.$$

In other words L^\pm are eigenvectors for operators $\text{ad } Z$ of adjoint representation of \mathfrak{sl}_2 [96, § VI.2].

Remark 3.7. The existence of such ladder operators follows from the general properties of Lie algebras if the element $X \in \mathfrak{sl}_2$ belongs to a *Cartan subalgebra*. This is the case for vectors Z and B , which are the only two non-isomorphic types of Cartan subalgebras in \mathfrak{sl}_2 . However the third case considered in this paper, the parabolic vector $B + Z/2$, does not belong to a Cartan subalgebra, yet a sort of ladder operators is still possible with dual number coefficients. Moreover, for the hyperbolic vector B , besides the standard ladder operators an additional pair with double number coefficients will also be described.

From the commutators (3.14) we deduce that L^+v_k are eigenvectors of \tilde{Z} as well:

$$(3.15) \quad \begin{aligned} \tilde{Z}(L^+v_k) &= (L^+\tilde{Z} + \lambda_+L^+)v_k = L^+(\tilde{Z}v_k) + \lambda_+L^+v_k = ikL^+v_k + \lambda_+L^+v_k \\ &= (ik + \lambda_+)L^+v_k. \end{aligned}$$

Thus action of ladder operators on respective eigenspaces can be visualised by the diagram:

$$(3.16) \quad \dots \xrightleftharpoons[L^-]{L^+} V_{ik-\lambda} \xrightleftharpoons[L^-]{L^+} V_{ik} \xrightleftharpoons[L^-]{L^+} V_{ik+\lambda} \xrightleftharpoons[L^-]{L^+} \dots$$

Assuming $L^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ from the relations (3.13) and defining condition (3.14) we obtain linear equations with unknown a , b and c :

$$c = 0, \quad 2a = \lambda_+b, \quad -2b = \lambda_+a.$$

The equations have a solution if and only if $\lambda_+^2 + 4 = 0$, and the raising/lowering operators are $L^\pm = \pm i\tilde{A} + \tilde{B}$.

3.3.2. Hyperbolic Ladder Operators. Consider the case $X = 2B$ of a generator of the subgroup A' (2.6). The subgroup is not compact and eigenvalues of the operator \tilde{B} can be arbitrary, however raising/lowering operators are still important [44, § II.1; 101, § 1.1]. We again seek a solution in the form $L_h^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ for the commutator $[2\tilde{B}, L_h^+] = \lambda L_h^+$. We will get the system:

$$4c = \lambda a, \quad b = 0, \quad a = \lambda c.$$

A solution exists if and only if $\lambda^2 = 4$. There are obvious values $\lambda = \pm 2$ with the ladder operators $L_h^\pm = \pm 2\tilde{A} + \tilde{Z}$, see [44, § II.1; 101, § 1.1]. Each indecomposable \mathfrak{sl}_2 -module is formed by a one-dimensional chain of eigenvalues with a transitive action of ladder operators.

Admitting double numbers we have an extra possibility to satisfy $\lambda^2 = 4$ with values $\lambda = \pm 2j$. Then there is an additional pair of hyperbolic ladder operators

$L_j^\pm = \pm 2j\tilde{A} + \tilde{Z}$, which shift eigenvectors in the “orthogonal” direction to the standard operators L_h^\pm . Therefore an indecomposable \mathfrak{sl}_2 -module can be parametrised by a two-dimensional lattice of eigenvalues on the double number plane, see Fig. 10

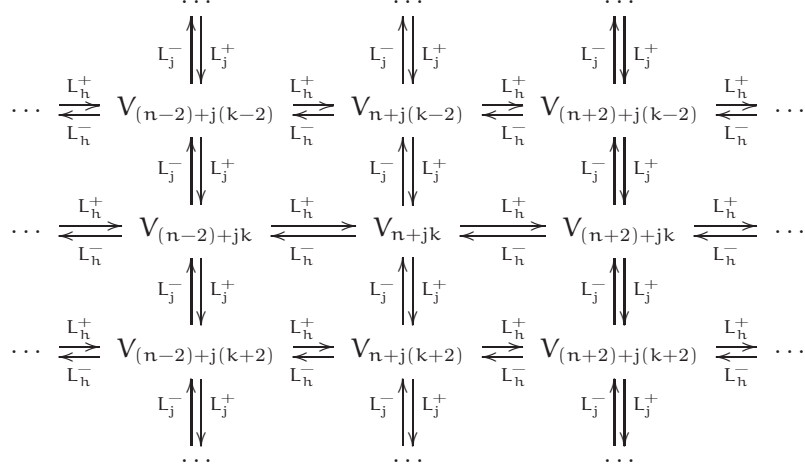


FIGURE 10. The action of hyperbolic ladder operators on a 2D lattice of eigenspaces. Operators L_h^\pm move the eigenvalues by 2, making shifts in the horizontal direction. Operators L_j^\pm change the eigenvalues by $2j$, shown as vertical shifts.

3.3.3. Parabolic Ladder Operators. Finally consider the case of a generator $X = -B + Z/2$ of the subgroup N' (2.5). According to the above procedure we get the equations:

$$b + 2c = \lambda a, \quad -a = \lambda b, \quad \frac{a}{2} = \lambda c,$$

which can be resolved if and only if $\lambda^2 = 0$. If we restrict ourselves with the only real (complex) root $\lambda = 0$, then the corresponding operators $L_p^\pm = -\tilde{B} + \tilde{Z}/2$ will not affect eigenvalues and thus are useless in the above context. However the dual number roots $\lambda = \pm \epsilon t$, $t \in \mathbb{R}$ lead to the operators $L_\epsilon^\pm = \pm \epsilon t \tilde{A} - \tilde{B} + \tilde{Z}/2$. These operators are suitable to build an \mathfrak{sl}_2 -modules with a one-dimensional chain of eigenvalues.

Remark 3.8. The following rôles of hypercomplex numbers are noteworthy:

- the introduction of complex numbers is a necessity for the *existence* of ladder operators in the elliptic case;
- in the parabolic case we need dual numbers to make ladder operators *useful*;
- in the hyperbolic case double numbers are not required neither for the existence or for the usability of ladder operators, but they do provide an enhancement.

We summarise the above consideration with a focus on the Principle of similarity and correspondence:

Proposition 3.9. *Let a vector $X \in \mathfrak{sl}_2$ generates the subgroup K , N' or A' , that is $X = Z$, $B - Z/2$, or B respectively. Let ι be the respective hypercomplex unit.*

Then raising/lowering operators L^\pm satisfying to the commutation relation:

$$[X, L^\pm] = \pm \iota L^\pm, \quad [L^-, L^+] = 2\iota X.$$

are:

$$L^\pm = \pm i\tilde{A} + \tilde{Y}.$$

Here $Y \in \mathfrak{sl}_2$ is a linear combination of B and Z with the properties:

- $Y = [A, X]$.
- $X = [A, Y]$.
- Killings form $K(X, Y)$ [55, § 6.2] vanishes.

Any of the above properties defines the vector $Y \in \text{span}\{B, Z\}$ up to a real constant factor.

The usability of the Principle of similarity and correspondence will be illustrated by more examples below.

4. COVARIANT TRANSFORM

A general group-theoretical construction [2, 18, 29, 32, 66, 92, 110] of *wavelets* (or *coherent state*) starts from an irreducible square integrable representation—in the proper sense or modulo a subgroup. Then a mother wavelet is chosen to be *admissible*. This leads to a wavelet transform which is an isometry to L_2 space with respect to the Haar measure on the group or (quasi)invariant measure on a homogeneous space.

The importance of the above situation shall not be diminished, however an exclusive restriction to such a setup is not necessary, in fact. Here is a classical example from complex analysis: the Hardy space $H_2(\mathbb{T})$ on the unit circle and Bergman spaces $B_2^n(\mathbb{D})$, $n \geq 2$ in the unit disk produce wavelets associated with representations ρ_1 and ρ_n of the group $SL_2(\mathbb{R})$ respectively [64]. While representations ρ_n , $n \geq 2$ are from square integrable discrete series, the mock discrete series representation ρ_1 is not square integrable [96, § VI.5; 117, § 8.4]. However it would be natural to treat the Hardy space in the same framework as Bergman ones. Some more examples will be presented below.

4.1. Extending Wavelet Transform. To make a sharp but still natural generalisation of wavelets we give the following definition.

Definition 4.1. [83] Let ρ be a representation of a group G in a space V and F be an operator from V to a space U . We define a *covariant transform* \mathcal{W} from V to the space $L(G, U)$ of U -valued functions on G by the formula:

$$(4.1) \quad \mathcal{W} : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, g \in G.$$

Operator F will be called *fiducial operator* in this context.

We borrow the name for operator F from fiducial vectors of Klauder and Skagerstam [92].

Remark 4.2. We do not require that fiducial operator F shall be linear. Sometimes the positive homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient, see Example 4.14.

Remark 4.3. Usefulness of the covariant transform is in the reverse proportion to the dimensionality of the space U . The covariant transform encodes properties of v in a function $\mathcal{W}v$ on G . For a low dimensional U this function can be ultimately investigated by means of harmonic analysis. Thus $\dim U = 1$ (scalar-valued functions) is the ideal case, however, it is unattainable sometimes, see Example 4.11 below. We may have to use a higher dimensions of U if the given group G is not rich enough.

As we will see below covariant transform is a close relative of wavelet transform. The name is chosen due to the following common property of both transformations.

Theorem 4.4. *The covariant transform (4.1) intertwines ρ and the left regular representation Λ on $L(G, \mathcal{U})$:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

Here Λ is defined as usual by:

$$(4.2) \quad \Lambda(g) : f(h) \mapsto f(g^{-1}h).$$

Proof. We have a calculation similar to wavelet transform [66, Prop. 2.6]. Take $u = \rho(g)v$ and calculate its covariant transform:

$$\begin{aligned} [\mathcal{W}(\rho(g)v)](h) &= [\mathcal{W}(\rho(g)v)](h) = F(\rho(h^{-1})\rho(g)v) \\ &= F(\rho((g^{-1}h)^{-1})v) \\ &= [\mathcal{W}v](g^{-1}h) \\ &= \Lambda(g)[\mathcal{W}v](h). \end{aligned}$$

□

The next result follows immediately:

Corollary 4.5. *The image space $\mathcal{W}(V)$ is invariant under the left shifts on G .*

Remark 4.6. A further generalisation of the covariant transform can be obtained if we relax the group structure. Consider, for example, a *cancellative semigroup* \mathbb{Z}_+ of non-negative integers. It has a linear presentation on the space of polynomials in a variable t defined by the action $m : t^n \mapsto t^{m+n}$ on the monomials. Application of a linear functional l , e.g. defined by an integration over a measure on the real line, produces *umbral calculus* $l(t^n) = c_n$, which has a magic efficiency in many areas, notably in combinatorics [67, 95]. In this direction we also find fruitful to expand the notion of an intertwining operator to a *token* [71].

4.2. Examples of Covariant Transform. In this Subsection we will provide several examples of covariant transforms. Some of them will be expanded in subsequent sections, however a detailed study of all aspects will not fit into the present work. We start from the classical example of the group-theoretical wavelet transform:

Example 4.7. Let V be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and ρ be a unitary representation of a group G in the space V . Let $F : V \rightarrow \mathbb{C}$ be a functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. The vector v_0 is oftenly called the *mother wavelet* in areas related to signal processing or the *vacuum state* in quantum framework.

Then the transformation (4.1) is the well-known expression for a *wavelet transform* [2, (7.48)] (or *representation coefficients*):

$$(4.3) \quad \mathcal{W} : v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \quad v \in V, g \in G.$$

The family of vectors $v_g = \rho(g)v_0$ is called *wavelets* or *coherent states*. In this case we obtain scalar valued functions on G , thus the fundamental rôle of this example is explained in Rem. 4.3.

This scheme is typically carried out for a square integrable representation ρ and v_0 being an admissible vector [2, 18, 29, 32, 110]. In this case the wavelet (covariant) transform is a map into the square integrable functions [26] with respect to the left Haar measure. The map becomes an isometry if v_0 is properly scaled.

However square integrable representations and admissible vectors does not cover all interesting cases.

Example 4.8. Let $G = \text{Aff}$ be the “ $ax + b$ ” (or *affine*) group [2, § 8.2]: the set of points (a, b) , $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ in the upper half-plane with the group law:

$$(4.4) \quad (a, b) * (a', b') = (aa', ab' + b)$$

and left invariant measure $a^{-2} da db$. Its isometric representation on $V = L_p(\mathbb{R})$ is given by the formula:

$$(4.5) \quad [\rho_p(g)f](x) = a^{\frac{1}{p}} f(ax + b), \quad \text{where } g^{-1} = (a, b).$$

We consider the operators $F_{\pm} : L_2(\mathbb{R}) \rightarrow \mathbb{C}$ defined by:

$$(4.6) \quad F_{\pm}(f) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t) dt}{x \mp i}.$$

Then the covariant transform (4.1) is the Cauchy integral from $L_p(\mathbb{R})$ to the space of functions $\hat{f}(a, b)$ such that $a^{-\frac{1}{p}} \hat{f}(a, b)$ is in the Hardy space in the upper/lower half-plane $H_p(\mathbb{R}_{\pm}^2)$. Although the representation (4.5) is square integrable for $p = 2$, the function $\frac{1}{x \pm i}$ used in (4.6) is not an admissible vacuum vector. Thus the complex analysis become decoupled from the traditional wavelet theory. As a result the application of wavelet theory shall rely on an extraneous mother wavelets [47].

Many important objects in complex analysis are generated by inadmissible mother wavelets like (4.6). For example, if $F : L_2(\mathbb{R}) \rightarrow \mathbb{C}$ is defined by $F : f \mapsto F_+ f + F_- f$ then the covariant transform (4.1) reduces to the *Poisson integral*. If $F : L_2(\mathbb{R}) \rightarrow \mathbb{C}^2$ is defined by $F : f \mapsto (F_+ f, F_- f)$ then the covariant transform (4.1) represents a function f on the real line as a jump:

$$(4.7) \quad f(z) = f_+(z) - f_-(z), \quad f_{\pm}(z) \in H_p(\mathbb{R}_{\pm}^2)$$

between functions analytic in the upper and the lower half-planes. This makes a decomposition of $L_2(\mathbb{R})$ into irreducible components of the representation (4.5). Another interesting but non-admissible vector is the *Gaussian* e^{-x^2} .

Example 4.9. For the group $G = SL_2(\mathbb{R})$ [96] let us consider the unitary representation ρ on the space of square integrable function $L_2(\mathbb{R}_+^2)$ on the upper half-plane through the Möbius transformations (1.1):

$$(4.8) \quad \rho(g) : f(z) \mapsto \frac{1}{(cz + d)^2} f\left(\frac{az + b}{cz + d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is a representation from the discrete series and $L_2(\mathbb{D})$ and irreducible invariant subspaces are parametrised by integers. Let F_k be the functional $L_2(\mathbb{R}_+^2) \rightarrow \mathbb{C}$ of pairing with the lowest/highest k -weight vector in the corresponding irreducible component (Bergman space) $B_k(\mathbb{R}_{\pm}^2)$, $k \geq 2$ of the discrete series [96, Ch. VI]. Then we can build an operator F from various F_k similarly to the previous Example. In particular, the jump representation (4.7) on the real line generalises to the representation of a square integrable function f on the upper half-plane as a sum

$$f(z) = \sum_k a_k f_k(z), \quad f_k \in B_n(\mathbb{R}_{\pm}^2)$$

for prescribed coefficients a_k and analytic functions f_k in question from different irreducible subspaces.

Covariant transform is also meaningful for principal and complementary series of representations of the group $SL_2(\mathbb{R})$, which are not square integrable [64].

Example 4.10. Let $G = SU(2) \times \text{Aff}$ be the Cartesian product of the groups $SU(2)$ of unitary rotations of \mathbb{C}^2 and the $ax + b$ group Aff . This group has a unitary linear representation on the space $L_2(\mathbb{R}, \mathbb{C}^2)$ of square-integrable (vector) \mathbb{C}^2 -valued functions by the formula:

$$\rho(g) \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} \alpha f_1(at + b) + \beta f_2(at + b) \\ a m m f_1(at + b) + \delta f_2(at + b) \end{pmatrix},$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \times (a, b) \in \text{SU}(2) \times \text{Aff}$. It is obvious that the vector Hardy space, that is functions with both components being analytic, is invariant under such action of G .

As a fiducial operator $F : L_2(\mathbb{R}, \mathbb{C}^2) \rightarrow \mathbb{C}$ we can take, cf. (4.6):

$$(4.9) \quad F \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f_1(t) dt}{x - i}.$$

Thus the image of the associated covariant transform is a subspace of scalar valued bounded functions on G . In this way we can transform (without a loss of information) vector-valued problems, e.g. matrix *Wiener–Hopf factorisation* [12], to scalar question of harmonic analysis on the group G .

Example 4.11. A straightforward generalisation of Ex. 4.7 is obtained if V is a Banach space and $F : V \rightarrow \mathbb{C}$ is an element of V^* . Then the covariant transform coincides with the construction of wavelets in Banach spaces [66].

Example 4.12. The next stage of generalisation is achieved if V is a Banach space and $F : V \rightarrow \mathbb{C}^n$ is a linear operator. Then the corresponding covariant transform is a map $\mathcal{W} : V \rightarrow L(G, \mathbb{C}^n)$. This is closely related to M.G. Krein’s works on *directing functionals* [93], see also *multiresolution wavelet analysis* [14], Clifford-valued Fock–Segal–Bargmann spaces [20] and [2, Thm. 7.3.1].

Example 4.13. Let F be a projector $L_p(\mathbb{R}) \rightarrow L_p(\mathbb{R})$ defined by the relation $(Ff)^\wedge(\lambda) = \chi(\lambda)\hat{f}(\lambda)$, where the hat denotes the Fourier transform and $\chi(\lambda)$ is the characteristic function of the set $[-2, -1] \cup [1, 2]$. Then the covariant transform $L_p(\mathbb{R}) \rightarrow C(\text{Aff}, L_p(\mathbb{R}))$ generated by the representation (4.5) of the affine group from F contains all information provided by the *Littlewood–Paley operator* [34, § 5.1.1].

Example 4.14. A step in a different direction is a consideration of non-linear operators. Take again the “ $ax + b$ ” group and its representation (4.5). We define F to be a homogeneous but non-linear functional $V \rightarrow \mathbb{R}_+$:

$$F(f) = \frac{1}{2} \int_{-1}^1 |f(x)| dx.$$

The covariant transform (4.1) becomes:

$$(4.10) \quad [\mathcal{W}_p f](a, b) = F(\rho_p(a, b)f) = \frac{1}{2} \int_{-1}^1 \left| a^{\frac{1}{p}} f(ax + b) \right| dx = a^{\frac{1}{p}} \frac{1}{2a} \int_{b-a}^{b+a} |f(x)| dx.$$

Obviously $M_f(b) = \max_a [\mathcal{W}_\infty f](a, b)$ coincides with the Hardy *maximal function*, which contains important information on the original function f . From the Cor. 4.5 we deduce that the operator $M : f \mapsto M_f$ intertwines ρ_p with itself $\rho_p M = M \rho_p$.

Of course, the full covariant transform (4.10) is even more detailed than M . For example, $\|f\| = \max_b [\mathcal{W}_\infty f](\frac{1}{2}, b)$ is the shift invariant norm [48].

Example 4.15. Let $V = L_c(\mathbb{R}^2)$ be the space of compactly supported bounded functions on the plane. We take F be the linear operator $V \rightarrow \mathbb{C}$ of integration over the real line:

$$F : f(x, y) \mapsto F(f) = \int_{\mathbb{R}} f(x, 0) dx.$$

Let G be the group of Euclidean motions of the plane represented by ρ on V by a change of variables. Then the wavelet transform $F(\rho(g)f)$ is the *Radon transform* [39].

4.3. Symbolic Calculi. There is a very important class of the covariant transforms which maps operators to functions. Among numerous sources we wish to single out works of Berezin [10, 11]. We start from the Berezin *covariant symbol*.

Example 4.16. Let a representation ρ of a group G act on a space X . Then there is an associated representation ρ_B of G on a space $V = B(X, Y)$ of linear operators $X \rightarrow Y$ defined by the identity [11, 66]:

$$(4.11) \quad (\rho_B(g)A)x = A(\rho(g^{-1})x), \quad x \in X, g \in G, A \in B(X, Y).$$

Following the Remark 4.3 we take F to be a functional $V \rightarrow \mathbb{C}$, for example F can be defined from a pair $x \in X, l \in Y^*$ by the expression $F : A \mapsto \langle Ax, l \rangle$. Then the covariant transform is:

$$\mathcal{W} : A \mapsto \hat{A}(g) = F(\rho_B(g)A).$$

This is an example of *covariant calculus* [10, 66].

There are several variants of the last Example which are of a separate interest.

Example 4.17. A modification of the previous construction is obtained if we have two groups G_1 and G_2 represented by ρ_1 and ρ_2 on X and Y^* respectively. Then we have a covariant transform $B(X, Y) \rightarrow L(G_1 \times G_2, \mathbb{C})$ defined by the formula:

$$\mathcal{W} : A \mapsto \hat{A}(g_1, g_2) = \langle A\rho_1(g_1)x, \rho_2(g_2)l \rangle.$$

This generalises the above *Berezin covariant calculi* [66].

Example 4.18. Let us restrict the previous example to the case when $X = Y$ is a Hilbert space, $\rho_1 = \rho_2 = \rho$ and $x = l$ with $\|x\| = 1$. Then the range of the covariant transform:

$$\mathcal{W} : A \mapsto \hat{A}(g) = \langle A\rho(g)x, \rho(g)x \rangle$$

is a subset of the *numerical range* of the operator A . As a function on a group $\hat{A}(g)$ provides a better description of A than the set of its values—numerical range.

Example 4.19. The group $SU(1, 1) \simeq SL_2(\mathbb{R})$ consists of 2×2 matrices of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with the unit determinant [96, § IX.1]. Let T be an operator with the spectral radius less than 1. Then the associated Möbius transformation

$$(4.12) \quad g : T \mapsto g \cdot T = \frac{\alpha T + \beta I}{\bar{\beta} T + \bar{\alpha} I}, \quad \text{where } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SL_2(\mathbb{R}),$$

produces a well-defined operator with the spectral radius less than 1 as well. Thus we have a representation of $SU(1, 1)$.

Let us introduce the *defect operators* $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$. For the fiducial operator $F = D_{T^*}$ the covariant transform is, cf. [115, § VI.1, (1.2)]:

$$[\mathcal{W}T](g) = F(g \cdot T) = -e^{i\Phi} \Theta_T(z) D_T, \quad \text{for } g = \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} \begin{pmatrix} 1 & -z \\ -\bar{z} & 1 \end{pmatrix},$$

where the *characteristic function* $\Theta_T(z)$ [115, § VI.1, (1.1)] is:

$$\Theta_T(z) = -T + D_{T^*} (I - zT^*)^{-1} z D_T.$$

Thus we approached the *functional model* of operators from the covariant transform. In accordance with Remark 4.3 the model is most fruitful for the case of operator $F = D_{T^*}$ being one-dimensional.

The intertwining property in the previous examples was obtained as a consequence of the general Prop. 4.4 about the covariant transform. However it may be worth to select it as a separate definition:

Definition 4.20. A *covariant calculus*, also known as *symbolic calculus*, is a map from operators to functions, which intertwines two representations of the same group in the respective spaces.

There is a dual class of covariant transforms acting in the opposite direction: from functions to operators. The prominent examples are the Berezin *contravariant symbol* [10, 66] and symbols of a *pseudodifferential operators* (PDO) [43, 66].

Example 4.21. The classical *Riesz–Dunford functional calculus* [27, § VII.3; 105, § IV.2] maps analytical functions on the unit disk to the linear operators, it is defined through the Cauchy-type formula with the resolvent. The calculus is an intertwining operator [75] between the Möbius transformations of the unit disk, cf. (5.22), and the actions (4.12) on operators from the Example 4.19. This topic will be developed in Subsection 6.1.

In line with the Defn. 4.20 we can directly define the corresponding calculus through the intertwining property [62, 75]:

Definition 4.22. A *contravariant calculus*, also known as *functional calculus*, is a map from functions to operators, which intertwines two representations of the same group in the respective spaces.

The duality between co- and contravariant calculi is the particular case of the duality between covariant transform and the inverse covariant transform defined in the next Subsection. In many cases a proper choice of spaces makes covariant and/or contravariant calculus a bijection between functions and operators. Subsequently only one form of calculus, either co- or contravariant, is defined explicitly, although both of them are there in fact.

4.4. Inverse Covariant Transform. An object invariant under the left action Λ (4.2) is called *left invariant*. For example, let L and L' be two left invariant spaces of functions on G . We say that a pairing $\langle \cdot, \cdot \rangle : L \times L' \rightarrow \mathbb{C}$ is *left invariant* if

$$(4.13) \quad \langle \Lambda(g)f, \Lambda(g)f' \rangle = \langle f, f' \rangle, \quad \text{for all } f \in L, f' \in L'.$$

Remark 4.23. (i) We do not require the pairing to be linear in general.

- (ii) If the pairing is invariant on space $L \times L'$ it is not necessarily invariant (or even defined) on the whole $C(G) \times C(G)$.
- (iii) In a more general setting we shall study an invariant pairing on a homogeneous spaces instead of the group. However due to length constraints we cannot consider it here beyond the Example 4.26.
- (iv) An invariant pairing on G can be obtained from an *invariant functional* l by the formula $\langle f_1, f_2 \rangle = l(f_1 \bar{f}_2)$.

For a representation ρ of G in V and $v_0 \in V$ we fix a function $w(g) = \rho(g)v_0$. We assume that the pairing can be extended in its second component to this V -valued functions, say, in the weak sense.

Definition 4.24. Let $\langle \cdot, \cdot \rangle$ be a left invariant pairing on $L \times L'$ as above, let ρ be a representation of G in a space V , we define the function $w(g) = \rho(g)v_0$ for $v_0 \in V$. The *inverse covariant transform* \mathcal{M} is a map $L \rightarrow V$ defined by the pairing:

$$(4.14) \quad \mathcal{M} : f \mapsto \langle f, w \rangle, \quad \text{where } f \in L.$$

Example 4.25. Let G be a group with a unitary square integrable representation ρ . An invariant pairing of two square integrable functions is obviously done by the integration over the Haar measure:

$$\langle f_1, f_2 \rangle = \int_G f_1(g) \bar{f}_2(g) dg.$$

For an admissible vector v_0 [2, Chap. 8; 26] the inverse covariant transform is known in this setup as a *reconstruction formula*.

Example 4.26. Let ρ be a square integrable representation of G modulo a subgroup $H \subset G$ and let $X = G/H$ be the corresponding homogeneous space with a quasi-invariant measure dx . Then integration over dx with an appropriate weight produces an invariant pairing. The inverse covariant transform is a more general version [2, (7.52)] of the *reconstruction formula* mentioned in the previous example.

Let ρ be not a square integrable representation (even modulo a subgroup) or let v_0 be inadmissible vector of a square integrable representation ρ . An invariant pairing in this case is not associated with an integration over any non singular invariant measure on G . In this case we have a *Hardy pairing*. The following example explains the name.

Example 4.27. Let G be the “ $ax + b$ ” group and its representation ρ (4.5) from Ex. 4.8. An invariant pairing on G , which is not generated by the Haar measure $a^{-2} da db$, is:

$$(4.15) \quad \langle f_1, f_2 \rangle = \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} f_1(a, b) \bar{f}_2(a, b) db.$$

For this pairing we can consider functions $\frac{1}{2\pi i(x+i)}$ or e^{-x^2} , which are not admissible vectors in the sense of square integrable representations. Then the inverse covariant transform provides an *integral resolutions* of the identity.

Similar pairings can be defined for other semi-direct products of two groups. We can also extend a Hardy pairing to a group, which has a subgroup with such a pairing.

Example 4.28. Let G be the group $SL_2(\mathbb{R})$ from the Ex. 4.9. Then the “ $ax+b$ ” group is a subgroup of $SL_2(\mathbb{R})$, moreover we can parametrise $SL_2(\mathbb{R})$ by triples (a, b, θ) , $\theta \in (-\pi, \pi]$ with the respective Haar measure [96, III.1(3)]. Then the Hardy pairing

$$(4.16) \quad \langle f_1, f_2 \rangle = \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} f_1(a, b, \theta) \bar{f}_2(a, b, \theta) db d\theta.$$

is invariant on $SL_2(\mathbb{R})$ as well. The corresponding inverse covariant transform provides even a finer resolution of the identity which is invariant under conformal mappings of the Lobachevsky half-plane.

5. ANALYTIC FUNCTIONS

We saw in the first section that an inspiring geometry of cycles can be recovered from the properties of $SL_2(\mathbb{R})$. In this section we consider a realisation of the function theory within Erlangen approach [64, 65, 68, 69]. The covariant transform will be our principal tool in this construction.

5.1. Induced Covariant Transform. The choice of a mother wavelet or fiducial operator F from Section 4.1 can significantly influence the behaviour of the covariant transform. Let G be a group and H be its closed subgroup with the corresponding homogeneous space $X = G/H$. Let ρ be a representation of G by operators on a space V , we denote by ρ_H the restriction of ρ to the subgroup H .

Definition 5.1. Let χ be a representation of the subgroup H in a space U and $F : V \rightarrow U$ be an intertwining operator between χ and the representation ρ_H :

$$(5.1) \quad F(\rho(h)v) = F(v)\chi(h), \quad \text{for all } h \in H, v \in V.$$

Then the covariant transform (4.1) generated by F is called the *induced covariant transform*

The following is the main motivating example.

Example 5.2. Consider the traditional wavelet transform as outlined in Ex. 4.7. Chose a vacuum vector v_0 to be a joint eigenvector for all operators $\rho(h)$, $h \in H$, that is $\rho(h)v_0 = \chi(h)v_0$, where $\chi(h)$ is a complex number depending of h . Then χ is obviously a character of H .

The image of wavelet transform (4.3) with such a mother wavelet will have a property:

$$\hat{v}(gh) = \langle v, \rho(gh)v_0 \rangle = \langle v, \rho(g)\chi(h)v_0 \rangle = \chi(h)\hat{v}(g).$$

Thus the wavelet transform is uniquely defined by cosets on the homogeneous space G/H . In this case we previously spoke about the *reduced wavelet transform* [65]. A representation ρ_0 is called *square integrable mod H* if the induced wavelet transform $[Wf_0](w)$ of the vacuum vector $f_0(x)$ is square integrable on X .

The image of induced covariant transform have the similar property:

$$(5.2) \quad \hat{v}(gh) = F(\rho((gh)^{-1})v) = F(\rho(h^{-1})\rho(g^{-1})v) = F(\rho(g^{-1})v)\chi(h^{-1}).$$

Thus it is enough to know the value of the covariant transform only at a single element in every coset G/H in order to reconstruct it for the entire group G by the representation χ . Since coherent states (wavelets) are now parametrised by points homogeneous space G/H they are referred sometimes as coherent states which are not connected to a group [91], however this is true only in a very narrow sense as explained above.

Example 5.3. To make it more specific we can consider the representation of $SL_2(\mathbb{R})$ defined on $L_2(\mathbb{R})$ by the formula, cf. (3.8):

$$\rho(g) : f(z) \mapsto \frac{1}{(cx + d)} f\left(\frac{ax + b}{cx + d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $K \subset SL_2(\mathbb{R})$ be the compact subgroup of matrices $h_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then for the fiducial operator F_{\pm} (4.6) we have $F_{\pm} \circ \rho(h_t) = e^{\mp it} F_{\pm}$. Thus we can consider the covariant transform only for points in the homogeneous space $SL_2(\mathbb{R})/K$, moreover this set can be naturally identified with the $ax + b$ group. Thus we do not obtain any advantage of extending the group in Ex. 4.8 from $ax + b$ to $SL_2(\mathbb{R})$ if we will be still using the fiducial operator F_{\pm} (4.6).

Functions on the group G , which have the property $\hat{v}(gh) = \hat{v}(g)\chi(h)$ (5.2), provide a space for the representation of G induced by the representation χ of the subgroup H . This explains the choice of the name for induced covariant transform.

Remark 5.4. Induced covariant transform uses the fiducial operator F which passes through the action of the subgroup H . This reduces information which we obtained from this transform in some cases.

There is also a simple connection between a covariant transform and right shifts:

Proposition 5.5. Let G be a Lie group and ρ be a representation of G in a space V . Let $[Wf](g) = F(\rho(g^{-1})f)$ be a covariant transform defined by the fiducial operator $F : V \rightarrow \mathcal{U}$. Then the right shift $[Wf](gg')$ by g' is the covariant transform $[W'f](g) = F'(\rho(g^{-1})f)$ defined by the fiducial operator $F' = F \circ \rho(g^{-1})$.

In other words the covariant transform intertwines right shifts on the group G with the associated action ρ_B (4.11) on fiducial operators.

Although the above result is obvious, its infinitesimal version has interesting consequences.

Corollary 5.6 ([84]). *Let G be a Lie group with a Lie algebra \mathfrak{g} and ρ be a smooth representation of G . We denote by $d\rho_B$ the derived representation of the associated representation ρ_B (4.11) on fiducial operators.*

Let a fiducial operator F be a null-solution, i.e. $AF = 0$, for the operator $A = \sum_j a_j d\rho_B^{X_j}$, where $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[Wf](g) = F(\rho(g^{-1})f)$ for any f satisfies:

$$DF(g) = 0, \quad \text{where} \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}.$$

Here \mathfrak{L}^{X_j} are the left invariant fields (Lie derivatives) on G corresponding to X_j .

Example 5.7. Consider the representation ρ (4.5) of the $ax + b$ group with the $p = 1$. Let A and N be the basis of the corresponding Lie algebra generating one-parameter subgroups $(e^t, 0)$ and $(0, t)$. Then the derived representations are:

$$[d\rho^A f](x) = f(x) + xf'(x), \quad [d\rho^N f](x) = f'(x).$$

The corresponding left invariant vector fields on $ax + b$ group are:

$$\mathfrak{L}^A = a\partial_a, \quad \mathfrak{L}^N = a\partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator $d\rho^A + id\rho^N = I + (x+i)\frac{d}{dx}$. Therefore the covariant transform with the fiducial operator F_+ (4.6) will consist with the null solutions to the operator $\mathfrak{L}^A - i\mathfrak{L}^N = -ia(\partial_b + i\partial_a)$, that is in the essence the Cauchy-Riemann operator in the upper half-plane.

There is a statement which extends the previous Corollary from differential operators to integro-differential ones. We will formulate it for the wavelets setting.

Corollary 5.8. *Let G be a group and ρ be a unitary representation of G , which can be extended to a vector space V of functions or distributions on G . Let a mother wavelet $w \in V'$ satisfy the equation*

$$\int_G a(g) \rho(g) w \, dg = 0,$$

for a fixed distribution $a(g) \in V$ and a (not necessarily invariant) measure dg . Then any wavelet transform $F(g) = Wf(g) = \langle f, \rho(g)w_0 \rangle$ obeys the condition:

$$DF = 0, \quad \text{where} \quad D = \int_G \bar{a}(g) R(g) \, dg,$$

with R being the right regular representation of G .

Clearly, the Corollary 5.6 is a particular case of the Corollary 5.8 with a distribution a , which is a combination of derivatives of Dirac's delta functions. The last Corollary will be illustrated at the end of Section 6.1.

Remark 5.9. We note that Corollaries 5.6 and 5.8 are true whenever we have an intertwining property between ρ with the right regular representation of G .

5.2. Induced Wavelet Transform and Cauchy Integral. We again use the general scheme from Subsection 3.2. The $ax + b$ group is isomorphic to a subgroups of $SL_2(\mathbb{R})$ consisting of the lower-triangular matrices:

$$F = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, a > 0 \right\}.$$

The corresponding homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/F$ is one-dimensional and can be parametrised by a real number. The natural projection $p : \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathbb{R}$ and its left inverse $s : \mathbb{R} \rightarrow \mathrm{SL}_2(\mathbb{R})$ can be defined as follows:

$$(5.3) \quad p : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{b}{d}, \quad s : u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Thus we calculate the corresponding map $r : \mathrm{SL}_2(\mathbb{R}) \rightarrow F$, see Subsection 2.1:

$$(5.4) \quad r : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}.$$

Therefore the action of $\mathrm{SL}_2(\mathbb{R})$ on the real line is exactly the Möbius map (1.1):

$$g : u \mapsto p(g^{-1} * s(u)) = \frac{au + b}{cu + d}, \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We also calculate that

$$r(g^{-1} * s(u)) = \begin{pmatrix} (cu + d)^{-1} & 0 \\ c & cu + d \end{pmatrix}.$$

To build an induced representation we need a character of the affine group. A generic character of F is a power of its diagonal element:

$$\rho_\kappa \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^\kappa.$$

Thus the corresponding realisation of induced representation (3.6) is:

$$(5.5) \quad \rho_\kappa(g) : f(u) \mapsto \frac{1}{(cu + d)^\kappa} f\left(\frac{au + b}{cu + d}\right) \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The only freedom remaining by the scheme is in a choice of a value of *number* κ and the corresponding functional space where our representation acts. At this point we have a wider choice of κ than it is usually assumed: it can belong to different hypercomplex systems.

One of the important properties which would be nice to have is the unitarity of the representation (5.5) with respect to the standard inner product:

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^2} f_1(u) \bar{f}_2(u) du.$$

A change of variables $x = \frac{au+b}{cu+d}$ in the integral suggests the following property is necessary and sufficient for that:

$$(5.6) \quad \kappa + \bar{\kappa} = 2.$$

A mother wavelet for an induced wavelet transform shall be an eigenvector for the action of a subgroup \tilde{H} of $\mathrm{SL}_2(\mathbb{R})$, see (5.1). Let us consider the most common case of $\tilde{H} = K$ and take the infinitesimal condition with the derived representation: $d\rho_n^Z w_0 = \lambda w_0$, since Z (3.12) is the generator of the subgroup K . In other word the restriction of w_0 to a K -orbit should be given by $e^{\lambda t}$ in the exponential coordinate t along the K -orbit. However we usually need its expression in other “more natural” coordinates. For example [86], an eigenvector of the derived representation of $d\rho_n^Z$ should satisfy the differential equation in the ordinary parameter $x \in \mathbb{R}$:

$$(5.7) \quad -\kappa x f(x) - f'(x)(1 + x^2) = \lambda f(x).$$

The equation does not have singular points, the general solution is globally defined (up to a constant factor) by:

$$(5.8) \quad w_{\lambda, \kappa}(x) = \frac{1}{(1 + x^2)^{\kappa/2}} \left(\frac{x - i}{x + i} \right)^{i\lambda/2} = \frac{(x - i)^{(i\lambda - \kappa)/2}}{(x + i)^{(i\lambda + \kappa)/2}}.$$

To avoid multivalent functions we need 2π -periodicity along the exponential coordinate on K . This implies that the parameter $m = -i\lambda$ is an integer. Therefore the solution becomes:

$$(5.9) \quad w_{m,\kappa}(x) = \frac{(x+i)^{(m-\kappa)/2}}{(x-i)^{(m+\kappa)/2}}.$$

The corresponding wavelets resemble the Cauchy kernel normalised to the invariant metric in the Lobachevsky half-plane:

$$w_{m,\kappa}(u, v; x) = \rho_\kappa^F(s(u, v)) w_{m,\kappa}(x) = v^{\kappa/2} \frac{(x-u+iv)^{(m-\kappa)/2}}{(x-u-iv)^{(m+\kappa)/2}}$$

Therefore the wavelet transform (4.3) from function on the real line to functions on the upper half-plane is:

$$\hat{f}(u, v) = \langle f, \rho_\kappa^F(u, v) w_{m,\kappa} \rangle = v^{\kappa/2} \int_{\mathbb{R}} f(x) \frac{(x-(u+iv))^{(m-\kappa)/2}}{(x-(u-iv))^{(m+\kappa)/2}} dx.$$

Introduction of a complex variable $z = u + iv$ allows to write it as:

$$(5.10) \quad \hat{f}(z) = (\Im z)^{\kappa/2} \int_{\mathbb{R}} f(x) \frac{(x-z)^{(m-\kappa)/2}}{(x-\bar{z})^{(m+\kappa)/2}} dx.$$

According to the general theory this wavelet transform intertwines representations ρ_κ^F (5.5) on the real line (induced by the character α^κ of the subgroup F) and ρ_m^K (3.8) on the upper half-plane (induced by the character e^{imt} of the subgroup K).

5.3. The Cauchy-Riemann (Dirac) and Laplace Operators. Ladder operators $L^\pm = \pm iA + B$ act by raising/lowering indexes of the K -eigenfunctions $w_{m,\kappa}$ (5.8), see Subsection 3.3. More explicitly [86]:

$$(5.11) \quad d\rho_\kappa^{L^\pm} : w_{m,\kappa} \mapsto -\frac{i}{2}(m \pm \kappa) w_{m \pm 2, \kappa}.$$

There are two possibilities here: $m \pm \kappa$ is zero for some m or not. In the first case the chain (5.11) of eigenfunction $w_{m,\kappa}$ terminates on one side under the transitive action (3.16) of the ladder operators; otherwise the chain is infinite in both directions. That is, the values $m = \mp \kappa$ and only those correspond to the maximal (minimal) weight function $w_{\mp \kappa, \kappa}(x) = \frac{1}{(x \pm i)^\kappa} \in L_2(\mathbb{R})$, which are annihilated by L^\pm :

$$(5.12) \quad d\rho_\kappa^{L^\pm} w_{\mp \kappa, \kappa} = (\pm i d\rho_\kappa^A + d\rho_\kappa^B) w_{\mp \kappa, \kappa} = 0.$$

By the Cor. 5.6 for the mother wavelets $w_{\mp \kappa, \kappa}$, which are annihilated by (5.12), the images of the respective wavelet transforms are null solutions to the left-invariant differential operator $D_\pm = \mathfrak{L}^{L^\pm}$:

$$(5.13) \quad D_\pm = \mp i \mathfrak{L}^A + \mathfrak{L}^B = -\frac{i\kappa}{2} + v(\partial_u \pm i\partial_v).$$

This is a conformal version of the Cauchy–Riemann equation. The second order conformal Laplace-type operators $\Delta_+ = \mathfrak{L}^{L^-} \mathfrak{L}^{L^+}$ and $\Delta_- = \mathfrak{L}^{L^+} \mathfrak{L}^{L^-}$ are:

$$(5.14) \quad \Delta_\pm = (v\partial_u - \frac{i\kappa}{2})^2 + v^2 \partial_v^2 \pm \frac{\kappa}{2}.$$

For the mother wavelets $w_{m,\kappa}$ in (5.12) such that $m = \mp \kappa$ the unitarity condition $\kappa + \bar{\kappa} = 2$, see (5.6), together with $m \in \mathbb{Z}$ implies $\kappa = \mp m = 1$. In such a case the wavelet transforms (5.10) are:

$$(5.15) \quad \hat{f}^+(z) = (\Im z)^{\frac{1}{2}} \int_{\mathbb{R}} \frac{f(x) dx}{x-z} \quad \text{and} \quad \hat{f}^-(z) = (\Im z)^{\frac{1}{2}} \int_{\mathbb{R}} \frac{f(x) dx}{x-\bar{z}},$$

for $w_{-1,1}$ and $w_{1,1}$ respectively. The first one is the Cauchy integral formula up to the factor $2\pi i \sqrt{z}$. Clearly, one integral is the complex conjugation of another. Moreover, the minimal/maximal weight cases can be intertwined by the following automorphism of the Lie algebra \mathfrak{sl}_2 :

$$A \rightarrow B, \quad B \rightarrow A, \quad Z \rightarrow -Z.$$

As explained before $\hat{f}^\pm(w)$ are null solutions to the operators D_\pm (5.13) and Δ_\pm (5.14). These transformations intertwine unitary equivalent representations on the real line and on the upper half-plane, thus they can be made unitary for proper spaces. This is the source of two faces of the Hardy spaces: they can be defined either as square-integrable on the real line with an analytic extension to the half-plane, or analytic on the half-plane with square-integrability on an infinitesimal displacement of the real line.

For the third possibility, $m \pm \kappa \neq 0$, there is no an operator spanned by the derived representation of the Lie algebra \mathfrak{sl}_2 which kills the mother wavelet $w_{m,\kappa}$. However the remarkable *Casimir operator* $C = Z^2 - 2(L^-L^+ + L^+L^-)$, which spans the centre of the universal enveloping algebra of \mathfrak{sl}_2 [96, § X.1; 117, § 8.1], produces a second order operator which does the job. Indeed from the identities (5.11) we get:

$$(5.16) \quad d\rho_\kappa^C w_{m,\kappa} = (2\kappa - \kappa^2)w_{m,\kappa}.$$

Thus we get $d\rho_\kappa^C w_{m,2} = 0$ for $\kappa = 2$ or 0 . The mother wavelet $w_{0,2}$ turns to be the *Poisson kernel* [34, Ex. 1.2.17]. The associated wavelet transform

$$(5.17) \quad \hat{f}(w) = \Im z \int_{\mathbb{R}} \frac{f(x) dz}{|x - z|^2}$$

consists of null solutions of the left-invariant second-order Laplacian, image of the Casimir operator, cf. (5.14):

$$\Delta(=: \mathfrak{L}^C) = v^2 \partial_u^2 + v^2 \partial_v^2.$$

Another integral formula producing solutions to this equation delivered by the mother wavelet $w_{m,0}$ with the value $\kappa = 0$ in (5.16):

$$(5.18) \quad \hat{f}(z) = \int_{\mathbb{R}} f(x) \left(\frac{x - z}{x - \bar{z}} \right)^{m/2} dx.$$

Furthermore, we can introduce higher order differential operators. The functions $w_{\mp 2m+1,1}$ are annihilated by n -th power of operator $d\rho_\kappa^{L^\pm}$ with $1 \leq m \leq n$. By the Cor. 5.6 the the image of wavelet transform (5.10) from a mother wavelet $\sum_1^n a_m w_{\mp 2m,1}$ will consist of null-solutions of the n -th power D_\pm^n of the conformal Cauchy–Riemann operator (5.13). They are a conformal flavour of *polyanalytic* functions [6].

We can similarly look for mother wavelets which are eigenvectors for other types of one dimensional subgroups. Our consideration of subgroup K is simplified by several facts:

- The parameter κ takes only complex values.
- The derived representation does not have singular points on the real line.

For both subgroups A' and N' this will not be true. The further consideration will be given in [86].

5.4. The Taylor Expansion. Consider an induced wavelet transform generated by a Lie group G , its representation ρ and a mother wavelet w which is an eigenvector of a one-dimensional subgroup $\tilde{H} \subset G$. Then by Prop. 5.5 the wavelet transform intertwines ρ with a representation $\rho^{\tilde{H}}$ induced by a character of \tilde{H} .

If the mother wavelet is itself in the domain of the induced wavelet transform then the chain (3.16) of \tilde{H} -eigenvectors w_m will be mapped to the similar chain of their images \hat{w}_m . The corresponding derived induced representation $d\rho^{\tilde{H}}$ produces ladder operators with the transitive action of the ladder operators on the chain of \hat{w}_m . Then the vector space of “formal power series”:

$$(5.19) \quad \hat{f}(z) = \sum_{m \in \mathbb{Z}} a_m \hat{w}_m(z)$$

is a module for the Lie algebra of the group G .

Coming back to the case of the group $G = \mathrm{SL}_2(\mathbb{R})$ and subgroup $\tilde{H} = K$. Images $\hat{w}_{m,1}$ of the eigenfunctions (5.9) under the Cauchy integral transform (5.15) are:

$$\hat{w}_{m,1}(z) = (\Im z)^{1/2} \frac{(z+i)^{(m-1)/2}}{(z-i)^{(m+1)/2}}.$$

They are eigenfunctions of the derived representation on the upper half-plane and the action of ladder operators is given by the same expressions (5.11). In particular, the \mathfrak{sl}_2 -module generated by $\hat{w}_{1,1}$ will be one-sided since this vector is annihilated by the lowering operator. Since the Cauchy integral produces an unitary intertwining operator between two representations we get the following variant of Taylor series:

$$\hat{f}(z) = \sum_{m=0}^{\infty} c_m \hat{w}_{m,1}(z), \quad \text{where} \quad c_m = \langle f, w_{m,1} \rangle.$$

For two other types of subgroups, representations and mother wavelets this scheme shall be suitably adapted and detailed study will be presented elsewhere [86].

5.5. Wavelet Transform in the Unit Disk and Other Domains. We can similarly construct an analytic function theories in unit disks, including parabolic and hyperbolic ones [82]. This can be done simply by an application of the *Cayley transform* to the function theories in the upper half-plane. Alternatively we can apply the full procedure for properly chosen groups and subgroups. We will briefly outline such a possibility here, see also [64].

Elements of $\mathrm{SL}_2(\mathbb{R})$ could be also represented by 2×2 -matrices with complex entries such that, cf. Example 4.21:

$$g = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

This realisations of $\mathrm{SL}_2(\mathbb{R})$ (or rather $\mathrm{SU}(2, \mathbb{C})$) is more suitable for function theory in the unit disk. It is obtained from the form, which we used before for the upper half-plane, by means of the Cayley transform [82, § 8.1].

We may identify the *unit disk* \mathbb{D} with the homogeneous space $\mathrm{SL}_2(\mathbb{R})/\mathbb{T}$ for the *unit circle* \mathbb{T} through the important decomposition $\mathrm{SL}_2(\mathbb{R}) \sim \mathbb{D} \times \mathbb{T}$ with $K = \mathbb{T}$ —the compact subgroup of $\mathrm{SL}_2(\mathbb{R})$:

$$(5.20) \quad \begin{aligned} \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} &= |\alpha| \begin{pmatrix} 1 & \bar{\beta}\bar{\alpha}^{-1} \\ \beta\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar{\alpha}}{|\alpha|} \end{pmatrix} \\ &= \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix}, \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \end{aligned}$$

where

$$x = \arg \alpha, \quad u = \bar{\beta} \bar{\alpha}^{-1}, \quad |u| < 1.$$

Each element $g \in \mathrm{SL}_2(\mathbb{R})$ acts by the linear-fractional transformation (the Möbius map) on \mathbb{D} and \mathbb{T} $H_2(\mathbb{T})$ as follows:

$$(5.21) \quad g : z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad \text{where} \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

In the decomposition (5.20) the first matrix on the right hand side acts by transformation (5.21) as an orthogonal rotation of \mathbb{T} or \mathbb{D} ; and the second one—by transitive family of maps of the unit disk onto itself.

The representation induced by a complex-valued character $\chi_k(z) = z^{-k}$ of \mathbb{T} according to the Section 3.2 is:

$$(5.22) \quad \rho_k(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^k} f\left(\frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}\right) \quad \text{where} \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

The representation ρ_1 is unitary on square-integrable functions and irreducible on the *Hardy space* on the unit circle.

We choose [66, 68] K -invariant function $v_0(z) \equiv 1$ to be a *vacuum vector*. Thus the associated *coherent states*

$$v(g, z) = \rho_1(g)v_0(z) = (u - z)^{-1}$$

are completely determined by the point on the unit disk $u = \bar{\beta} \bar{\alpha}^{-1}$. The family of coherent states considered as a function of both u and z is obviously the *Cauchy kernel* [64]. The *wavelet transform* [64, 66] $\mathcal{W} : L_2(\mathbb{T}) \rightarrow H_2(\mathbb{D}) : f(z) \mapsto \mathcal{W}f(g) = \langle f, v_g \rangle$ is the *Cauchy integral*:

$$(5.23) \quad \mathcal{W}f(u) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{1}{u - z} dz.$$

This approach can be extended to arbitrary connected simply-connected domain. Indeed, it is known that Möbius maps is the whole group of biholomorphic automorphisms of the unit disk or upper half-plane. Thus we can state the following corollary from the *Riemann mapping theorem*:

Corollary 5.10. *The group of biholomorphic automorphisms of a connected simply connected domain with at least two points on its boundary is isomorphic to $\mathrm{SL}_2(\mathbb{R})$.*

If a domain is non-simply connected, then the group of its biholomorphic mapping can be trivial [9, 102]. However we may look for a rich group acting on function spaces rather than on geometric sets. Let a connected non-simply connected domain D be bounded by a finite collection of non-intersecting contours amma_i , $i = 1, \dots, n$. For each amma_i consider the isomorphic image G_i of the $\mathrm{SL}_2(\mathbb{R})$ group which is defined by the Corollary 5.10. Then define the group $G = G_1 \times G_2 \times \dots \times G_n$ and its action on $L_2(\partial D) = L_2(\text{amma}_1) \oplus L_2(\text{amma}_2) \oplus \dots \oplus L_2(\text{amma}_n)$ through the Moebius action of G_i on $L_2(\text{amma}_i)$.

Example 5.11. Consider an *annulus* defined by $r < |z| < R$. It is bounded by two circles: $\text{amma}_1 = \{z : |z| = r\}$ and $\text{amma}_2 = \{z : |z| = R\}$. For amma_1 the Möbius action of $\mathrm{SL}_2(\mathbb{R})$ is

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : z \mapsto \frac{\alpha z + \bar{\beta}/r}{\beta z/r + \bar{\alpha}}, \quad \text{where} \quad |\alpha|^2 - |\beta|^2 = 1,$$

with the respective action on amma_2 . Those action can be linearised in the spaces $L_2(\text{amma}_1)$ and $L_2(\text{amma}_2)$. If we consider a subrepresentation reduced to analytic function on the annulus, then one copy of $\mathrm{SL}_2(\mathbb{R})$ will act on the part of functions analytic outside of amma_1 and another copy—on the part of functions analytic inside of amma_2 .

Thus all classical objects of complex analysis (the Cauchy-Riemann equation, the Taylor series, the Bergman space, etc.) for a rather generic domain D can be also obtained from suitable representations similarly to the case of the upper half-plane [64, 68].

6. COVARIANT AND CONTRAVARIANT CALCULI

United in the trinity functional calculus, spectrum, and spectral mapping theorem play the exceptional rôle in functional analysis and could not be substituted by anything else. Many traditional definitions of functional calculus are covered by the following rigid template based on the *algebra homomorphism* property:

Definition 6.1. An *functional calculus* for an element $a \in \mathfrak{A}$ is a continuous linear mapping $\Phi : \mathcal{A} \rightarrow \mathfrak{A}$ such that

- (i) Φ is a unital *algebra homomorphism*

$$\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g).$$

- (ii) There is an initialisation condition: $\Phi[v_0] = a$ for a fixed function v_0 , e.g. $v_0(z) = z$.

The most typical definition of the spectrum is seemingly independent and uses the important notion of resolvent:

Definition 6.2. A *resolvent* of element $a \in \mathfrak{A}$ is the function $R(\lambda) = (a - \lambda e)^{-1}$, which is the image under Φ of the Cauchy kernel $(z - \lambda)^{-1}$.

A *spectrum* of $a \in \mathfrak{A}$ is the set $\text{sp } a$ of singular points of its resolvent $R(\lambda)$.

Then the following important theorem links spectrum and functional calculus together.

Theorem 6.3 (Spectral Mapping). *For a function f suitable for the functional calculus:*

$$(6.1) \quad f(\text{sp } a) = \text{sp } f(a).$$

However the power of the classic spectral theory rapidly decreases if we move beyond the study of one normal operator (e.g. for quasinilpotent ones) and is virtually nil if we consider several non-commuting ones. Sometimes these severe limitations are seen to be irresistible and alternative constructions, i.e. model theory cf. Example 4.19 and [105], were developed.

Yet the spectral theory can be revived from a fresh start. While three components—functional calculus, spectrum, and spectral mapping theorem—are highly interdependent in various ways we will nevertheless arrange them as follows:

- (i) Functional calculus is an *original* notion defined in some independent terms;
- (ii) Spectrum (or more specifically *contravariant spectrum*) (or spectral decomposition) is derived from previously defined functional calculus as its *support* (in some appropriate sense);
- (iii) Spectral mapping theorem then should drop out naturally in the form (6.1) or some its variation.

Thus the entire scheme depends from the notion of the functional calculus and our ability to escape limitations of Definition 6.1. The first known to the present author definition of functional calculus not linked to algebra homomorphism property was the Weyl functional calculus defined by an integral formula [3]. Then its intertwining property with affine transformations of Euclidean space was proved as a theorem. However it seems to be the only “non-homomorphism” calculus for decades.

The different approach to whole range of calculi was given in [62] and developed in [66,73,75,84] in terms of *intertwining operators* for group representations. It was initially targeted for several non-commuting operators because no non-trivial algebra homomorphism is possible with a commutative algebra of function in this case. However it emerged later that the new definition is a useful replacement for classical one across all range of problems.

In the following Subsections we will support the last claim by consideration of the simple known problem: characterisation a $n \times n$ matrix up to similarity. Even that “freshman” question could be only sorted out by the classical spectral theory for a small set of diagonalisable matrices. Our solution in terms of new spectrum will be full and thus unavoidably coincides with one given by the Jordan normal form of matrices. Other more difficult questions are the subject of ongoing research.

6.1. Intertwining Group Actions on Functions and Operators. Any functional calculus uses properties of *functions* to model properties of *operators*. Thus changing our viewpoint on functions, as was done in Section 5, we could get another approach to operators. The two main possibilities are encoded in Definitions 4.20 and 4.22: we can assign a certain function to the given operator or wise verse. Here we consider the second possibility and treat the first in the Subsection 6.4.

The representation ρ_1 (5.22) is unitary irreducible when acts on the Hardy space H_2 . Consequently we have one more reason to abolish the template definition 6.1: H_2 is *not* an algebra. Instead we replace the *homomorphism property* by a *symmetric covariance*:

Definition 6.4 ([62]). An *contravariant analytic calculus* for an element $a \in \mathfrak{A}$ and an \mathfrak{A} -module M is a *continuous linear mapping* $\Phi : A(\mathbb{D}) \rightarrow A(\mathbb{D}, M)$ such that

- (i) Φ is an *intertwining operator*

$$\Phi \rho_1 = \rho_a \Phi$$

between two representations of the $SL_2(\mathbb{R})$ group ρ_1 (5.22) and ρ_a defined below in (6.4).

- (ii) There is an initialisation condition: $\Phi[v_0] = m$ for $v_0(z) \equiv 1$ and $m \in M$, where M is a left \mathfrak{A} -module.

Note that our functional calculus released from the homomorphism condition can take value in any left \mathfrak{A} -module M , which however could be \mathfrak{A} itself if suitable. This add much flexibility to our construction.

The earliest functional calculus, which is *not* an algebraic homomorphism, was the Weyl functional calculus and was defined just by an integral formula as an operator valued distribution [3]. In that paper (joint) spectrum was defined as support of the Weyl calculus, i.e. as the set of point where this operator valued distribution does not vanish. We also define the spectrum as a support of functional calculus, but due to our Definition 6.4 it will means the set of non-vanishing intertwining operators with primary subrepresentations.

Definition 6.5. A corresponding *spectrum* of $a \in \mathfrak{A}$ is the *support* of the functional calculus Φ , i.e. the collection of intertwining operators of ρ_a with *primary representations* [55, § 8.3].

More variations of contravariant functional calculi are obtained from other groups and their representations [62,66,73,75,84].

A simple but important observation is that the Möbius transformations (1.1) can be easily extended to any Banach algebra. Let \mathfrak{A} be a Banach algebra with the unit

e , an element $a \in \mathfrak{A}$ with $\|a\| < 1$ be fixed, then

$$(6.2) \quad g : a \mapsto g \cdot a = (\bar{\alpha}a - \bar{\beta}e)(\alpha e - \beta a)^{-1}, \quad g \in \mathrm{SL}_2(\mathbb{R})$$

is a well defined $\mathrm{SL}_2(\mathbb{R})$ action on a subset $\mathbb{A} = \{g \cdot a \mid g \in \mathrm{SL}_2(\mathbb{R})\} \subset \mathfrak{A}$, i.e. \mathbb{A} is a $\mathrm{SL}_2(\mathbb{R})$ -homogeneous space. Let us define the *resolvent* function $R(g, a) : \mathbb{A} \rightarrow \mathfrak{A}$:

$$R(g, a) = (\alpha e - \beta a)^{-1}$$

then

$$(6.3) \quad R(g_1, a)R(g_2, g_1^{-1}a) = R(g_1g_2, a).$$

The last identity is well known in representation theory [55, § 13.2(10)] and is a key ingredient of *induced representations*. Thus we can again linearise (6.2), cf. (5.22), in the space of continuous functions $C(\mathbb{A}, M)$ with values in a left \mathfrak{A} -module M , e.g. $M = \mathfrak{A}$:

$$(6.4) \quad \begin{aligned} \rho_a(g_1) : f(g^{-1} \cdot a) &\mapsto R(g_1^{-1}g^{-1}, a)f(g_1^{-1}g^{-1} \cdot a) \\ &= (\alpha'e - \beta'a)^{-1} f\left(\frac{\bar{\alpha}' \cdot a - \bar{\beta}'e}{\alpha'e - \beta'a}\right). \end{aligned}$$

For any $m \in M$ we can define a K -invariant *vacuum vector* as $v_m(g^{-1} \cdot a) = m \otimes v_0(g^{-1} \cdot a) \in C(\mathbb{A}, M)$. It generates the associated with v_m family of *coherent states* $v_m(u, a) = (ue - a)^{-1}m$, where $u \in \mathbb{D}$.

The *wavelet transform* defined by the same common formula based on coherent states (cf. (5.23)):

$$(6.5) \quad \mathcal{W}_m f(g) = \langle f, \rho_a(g)v_m \rangle,$$

is a version of Cauchy integral, which maps $L_2(\mathbb{A})$ to $C(\mathrm{SL}_2(\mathbb{R}), M)$. It is closely related (but not identical!) to the Riesz-Dunford functional calculus: the traditional functional calculus is given by the case:

$$\Phi : f \mapsto \mathcal{W}_m f(0) \quad \text{for } M = \mathfrak{A} \text{ and } m = e.$$

The both conditions—the intertwining property and initial value—required by Definition 6.4 easily follows from our construction. Finally, we wish to provide an example of application of the Corollary 5.8.

Example 6.6. Let a be an operator and ϕ be a function which annihilates it, i.e. $\phi(a) = 0$. For example, if a is a matrix ϕ can be its minimal polynomial. From the integral representation of the contravariant calculus on $G = \mathrm{SL}_2(\mathbb{R})$ we can rewrite the annihilation property like this:

$$\int_G \phi(g)R(g, a) dg = 0.$$

Then the vector-valued function $[\mathcal{W}_m f](g)$ defined by (6.5) shall satisfy to the following condition:

$$\int_G \phi(g') [\mathcal{W}_m f](gg') dg' = 0$$

due to the Corollary 5.8.

6.2. Jet Bundles and Prolongations of ρ_1 . Spectrum was defined in 6.5 as the *support* of our functional calculus. To elaborate its meaning we need the notion of a *prolongation* of representations introduced by S. Lie, see [106, 107] for a detailed exposition.

Definition 6.7. [107, Chap. 4] Two holomorphic functions have n th order contact in a point if their value and their first n derivatives agree at that point, in other words their Taylor expansions are the same in first $n + 1$ terms.

A point $(z, u^{(n)}) = (z, u, u_1, \dots, u_n)$ of the *jet space* $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having n th contact at the point z with the polynomial:

$$(6.6) \quad p_n(w) = u_n \frac{(w-z)^n}{n!} + \dots + u_1 \frac{(w-z)}{1!} + u.$$

For a fixed n each holomorphic function $f : \mathbb{D} \rightarrow \mathbb{C}$ has n th prolongation (or n -jet) $j_n f : \mathbb{D} \rightarrow \mathbb{C}^{n+1}$:

$$(6.7) \quad j_n f(z) = (f(z), f'(z), \dots, f^{(n)}(z)).$$

The graph $\text{amma}_f^{(n)}$ of $j_n f$ is a submanifold of \mathbb{J}^n which is section of the *jet bundle* over \mathbb{D} with a fibre \mathbb{C}^{n+1} . We also introduce a notation J_n for the map $J_n : f \mapsto \text{amma}_f^{(n)}$ of a holomorphic f to the graph $\text{amma}_f^{(n)}$ of its n -jet $j_n f(z)$ (6.7).

One can prolong any map of functions $\psi : f(z) \mapsto [\psi f](z)$ to a map $\psi^{(n)}$ of n -jets by the formula

$$(6.8) \quad \psi^{(n)}(J_n f) = J_n(\psi f).$$

For example such a prolongation $\rho_1^{(n)}$ of the representation ρ_1 of the group $\text{SL}_2(\mathbb{R})$ in $H_2(\mathbb{D})$ (as any other representation of a Lie group [107]) will be again a representation of $\text{SL}_2(\mathbb{R})$. Equivalently we can say that J_n intertwines ρ_1 and $\rho_1^{(n)}$:

$$J_n \rho_1(g) = \rho_1^{(n)}(g) J_n \quad \text{for all } g \in \text{SL}_2(\mathbb{R}).$$

Of course, the representation $\rho_1^{(n)}$ is not irreducible: any jet subspace \mathbb{J}^k , $0 \leq k \leq n$ is $\rho_1^{(n)}$ -invariant subspace of \mathbb{J}^n . However the representations $\rho_1^{(n)}$ are *primary* [55, § 8.3] in the sense that they are not sums of two subrepresentations.

The following statement explains why jet spaces appeared in our study of functional calculus.

Proposition 6.8. *Let matrix a be a Jordan block of a length k with the eigenvalue $\lambda = 0$, and m be its root vector of order k , i.e. $a^{k-1}m \neq a^k m = 0$. Then the restriction of ρ_a on the subspace generated by v_m is equivalent to the representation ρ_1^k .*

6.3. Spectrum and Spectral Mapping Theorem. Now we are prepared to describe a spectrum of a matrix. Since the functional calculus is an intertwining operator its support is a decomposition into intertwining operators with primary representations (we could not expect generally that these primary subrepresentations are irreducible).

Recall the transitive on \mathbb{D} group of inner automorphisms of $\text{SL}_2(\mathbb{R})$, which can send any $\lambda \in \mathbb{D}$ to 0 and are actually parametrised by such a λ . This group extends Proposition 6.8 to the complete characterisation of ρ_a for matrices.

Proposition 6.9. *Representation ρ_a is equivalent to a direct sum of the prolongations $\rho_1^{(k)}$ of ρ_1 in the k th jet space \mathbb{J}^k intertwined with inner automorphisms. Consequently the spectrum of a (defined via the functional calculus $\Phi = \mathcal{W}_m$) labelled exactly by n pairs of numbers (λ_i, k_i) , $\lambda_i \in \mathbb{D}$, $k_i \in \mathbb{Z}_+$, $1 \leq i \leq n$ some of whom could coincide.*

Obviously this spectral theory is a fancy restatement of the *Jordan normal form* of matrices.

Example 6.10. Let $J_k(\lambda)$ denote the Jordan block of the length k for the eigenvalue λ . In Fig. 11 there are two pictures of the spectrum for the matrix

$$a = J_3(\lambda_1) \oplus J_4(\lambda_2) \oplus J_1(\lambda_3) \oplus J_2(\lambda_4),$$

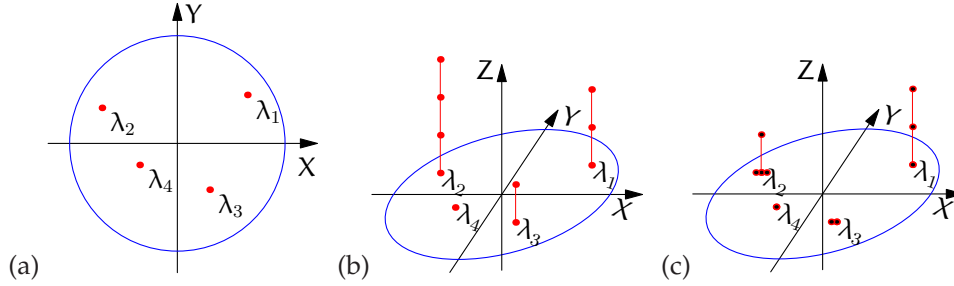


FIGURE 11. Classical spectrum of the matrix from the Ex. 6.10 is shown at (a). Contravariant spectrum of the same matrix in the jet space is drawn at (b). The image of the contravariant spectrum under the map from Ex. 6.12 is presented at (c).

where

$$\lambda_1 = \frac{3}{4}e^{i\pi/4}, \quad \lambda_2 = \frac{2}{3}e^{i5\pi/6}, \quad \lambda_3 = \frac{2}{5}e^{-i3\pi/4}, \quad \lambda_4 = \frac{3}{5}e^{-i\pi/3}.$$

Part (a) represents the conventional two-dimensional image of the spectrum, i.e. eigenvalues of α , and (b) describes spectrum $\text{sp } \alpha$ arising from the wavelet construction. The first image did not allow to distinguish α from many other essentially different matrices, e.g. the diagonal matrix

$$\text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

which even have a different dimensionality. At the same time Fig. 11(b) completely characterise α up to a similarity. Note that each point of $\text{sp } \alpha$ in Fig. 11(b) corresponds to a particular root vector, which spans a primary subrepresentation.

As was mentioned in the beginning of this section a reasonable spectrum should be linked to the corresponding functional calculus by an appropriate spectral mapping theorem. The new version of spectrum is based on prolongation of ρ_1 into jet spaces (see Section 6.2). Naturally a correct version of spectral mapping theorem should also operate in jet spaces.

Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic map, let us define its action on functions $[\phi_* f](z) = f(\phi(z))$. According to the general formula (6.8) we can define the prolongation $\phi_*^{(n)}$ onto the jet space \mathbb{J}^n . Its associated action $\rho_1^k \phi_*^{(n)} = \phi_*^{(n)} \rho_1^n$ on the pairs (λ, k) is given by the formula:

$$(6.9) \quad \phi_*^{(n)}(\lambda, k) = \left(\phi(\lambda), \left[\frac{k}{\deg_\lambda \phi} \right] \right),$$

where $\deg_\lambda \phi$ denotes the degree of zero of the function $\phi(z) - \phi(\lambda)$ at the point $z = \lambda$ and $[x]$ denotes the integer part of x .

Theorem 6.11 (Spectral mapping). *Let ϕ be a holomorphic mapping $\phi : \mathbb{D} \rightarrow \mathbb{D}$ and its prolonged action $\phi_*^{(n)}$ defined by (6.9), then*

$$\text{sp } \phi(\alpha) = \phi_*^{(n)} \text{sp } \alpha.$$

The explicit expression of (6.9) for $\phi_*^{(n)}$, which involves derivatives of ϕ upto n th order, is known, see for example [41, Thm. 6.2.25], but was not recognised before as form of spectral mapping.

Example 6.12. Let us continue with Example 6.10. Let ϕ map all four eigenvalues $\lambda_1, \dots, \lambda_4$ of the matrix α into themselves. Then Fig. 11(a) will represent the classical spectrum of $\phi(\alpha)$ as well as α .

However Fig. 11(c) shows mapping of the new spectrum for the case ϕ has orders of zeros at these points as follows: the order 1 at λ_1 , exactly the order 3 at λ_2 , an order at least 2 at λ_3 , and finally any order at λ_4 .

6.4. Functional Model and Spectral Distance. Let a be a matrix and $\mu(z)$ be its minimal polynomial:

$$\mu_a(z) = (z - \lambda_1)^{m_1} \cdot \dots \cdot (z - \lambda_n)^{m_n}.$$

If all eigenvalues λ_i of a (i.e. all roots of $\mu(z)$) belong to the unit disk we can consider the respective Blaschke product

$$B_a(z) = \prod_{i=1}^n \left(\frac{z - \lambda_i}{1 - \overline{\lambda_i}z} \right)^{m_i},$$

such that its numerator coincides with the minimal polynomial $\mu(z)$. Moreover, for an unimodular z we have $B_a(z) = \mu_a(z) \overline{\mu_a^{-1}(z)} z^{-m}$, where $m = m_1 + \dots + m_n$. We also have the following covariance property:

Proposition 6.13. *The above correspondence $a \mapsto B_a$ intertwines the $SL_2(\mathbb{R})$ action (6.2) on the matrices with the action (5.22) with $k = 0$ on functions.*

The result follows from the observation that every elementary product $\frac{z - \lambda_i}{1 - \overline{\lambda_i}z}$ is the Moebius transformation of z with the matrix $\begin{pmatrix} 1 & -\lambda_i \\ -\overline{\lambda_i} & 1 \end{pmatrix}$. Thus the correspondence $a \mapsto B_a(z)$ is a covariant (symbolic) calculus in the sense of the Defn. 4.20. See also the Example 4.19.

The Jordan normal form of a matrix provides a description, which is equivalent to its contravariant spectrum. From various viewpoints, e.g. numerical approximations, it is worth to consider its stability under a perturbation. It is easy to see, that an arbitrarily small disturbance breaks the Jordan structure of a matrix. However, the result of random small perturbation will not be random, its nature is described by the following remarkable theorem:

Theorem 6.14 (Lidskii [99], see also [103]). *Let J_n be a Jordan block of a length $n > 1$ with zero eigenvalues and K be an arbitrary matrix. Then eigenvalues of the perturbed matrix $J_n + \varepsilon^n K$ admit the expansion*

$$\lambda_j = \varepsilon \xi^{1/n} + o(\varepsilon), \quad j = 1, \dots, n,$$

where $\xi^{1/n}$ represents all n -th complex roots of certain $\xi \in \mathbb{C}$.

The left picture in Fig. 12 presents a perturbation of a Jordan block J_{100} by a random matrix. Perturbed eigenvalues are close to vertices of a right polygon with 100 vertices. Those regular arrangements occur despite of the fact that eigenvalues of the matrix K are dispersed through the unit disk (the right picture in Fig. 12). In a sense it is rather the Jordan block regularises eigenvalues of K than K perturbs the eigenvalue of the Jordan block.

Although the Jordan structure itself is extremely fragile, it still can be easily guessed from a perturbed eigenvalues. Thus there exists a certain characterisation of matrices which is stable under small perturbations. We will describe a sense, in which the covariant spectrum of the matrix $J_n + \varepsilon^n K$ is stable for small ε . For this we introduce the covariant version of spectral distances motivated by the functional model. Our definition is different from other types known in the literature [120, Ch. 5].

Definition 6.15. Let a and b be two matrices with all their eigenvalues sitting inside of the unit disk and $B_a(z)$ and $B_b(z)$ be respective Blaschke products as

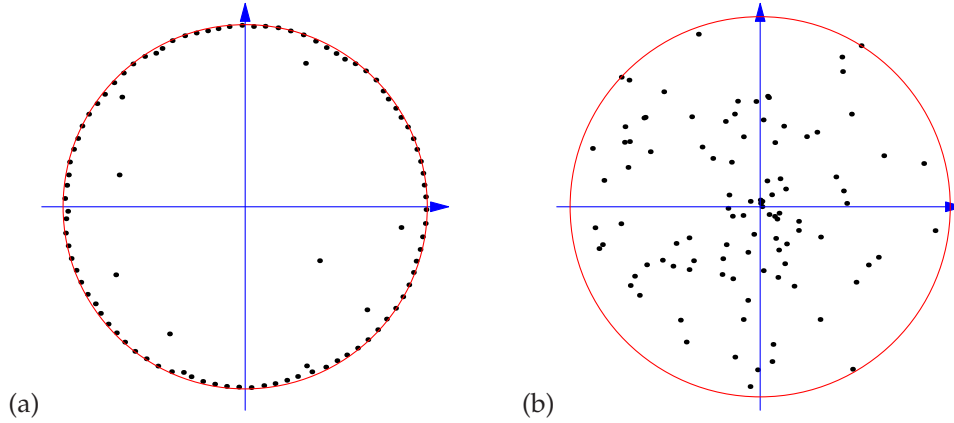


FIGURE 12. Perturbation of the Jordan block's spectrum: (a) The spectrum of the perturbation $J_{100} + \varepsilon^{100}K$ of the Jordan block J_{100} by a random matrix K . (b) The spectrum of the random matrix K .

defined above. The (covariant) spectral distance $d(a, b)$ between a and b is equal to the distance $\|B_a - B_b\|_2$ between $B_a(z)$ and $B_b(z)$ in the Hardy space on the unit circle.

Since the spectral distance is defined through the distance in H_2 all standard axioms of a distance are automatically satisfied. For a Blaschke products we have $|B_a(z)| = 1$ if $|z| = 1$, thus $\|B_a\|_p = 1$ in any L_p on the unit circle. Therefore an alternative expression for the spectral distance is:

$$d(a, b) = 2(1 - \langle B_a, B_b \rangle).$$

In particular, we always have $0 \leq d(a, b) \leq 2$. We get an obvious consequence of Prop. 6.13, which justifies the name of the covariant spectral distance:

Corollary 6.16. *For any $g \in \text{SL}_2(\mathbb{R})$ we have $d(a, b) = d(g \cdot a, g \cdot b)$, where \cdot denotes the Möbius action (6.2).*

An important property of the covariant spectral distance is its stability under small perturbations.

Theorem 6.17. *For $n = 2$ let $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ be eigenvalues of the matrix $J_2 + \varepsilon^2 \cdot K$ for some matrix K . Then*

$$(6.10) \quad |\lambda_1(\varepsilon)| + |\lambda_2(\varepsilon)| = O(\varepsilon), \quad \text{however} \quad |\lambda_1(\varepsilon) + \lambda_2(\varepsilon)| = O(\varepsilon^2).$$

The spectral distance from the 1-jet at 0 to two 0-jets at points λ_1 and λ_2 bounded only by the first condition in (6.10) is $O(\varepsilon^2)$. However the spectral distance between J_2 and $J_2 + \varepsilon^2 \cdot K$ is $O(\varepsilon^4)$.

In other words, a matrix with eigenvalues satisfying to the Lidskii condition from the Thm. 6.14 is much closer to the Jordan block J_2 than a generic one with eigenvalues of the same order. Thus the covariant spectral distance is more stable under perturbation than magnitude of eigenvalues. For $n = 2$ a proof can be forced through a direct calculation. We also conjecture that the similar statement is true for any $n \geq 2$.

6.5. Covariant Pencils of Operators. Let H be a real Hilbert space, possibly of finite dimensionality. For bounded linear operators A and B consider the *generalised eigenvalue problem*, that is finding a scalar λ and a vector $x \in H$ such that:

$$(6.11) \quad Ax = \lambda Bx \quad \text{or equivalently} \quad (A - \lambda B)x = 0.$$

The standard eigenvalue problem corresponds to the case $B = I$, moreover for an invertible B the generalised problem can be reduced to the standard one for the operator $B^{-1}A$. Thus it is sensible to introduce the equivalence relation on the pairs of operators:

$$(6.12) \quad (A, B) \sim (DA, DB) \quad \text{for any invertible operator } D.$$

We may treat the pair (A, B) as a column vector $\begin{pmatrix} A \\ B \end{pmatrix}$. Then there is an action of the $SL_2(\mathbb{R})$ group on the pairs:

$$(6.13) \quad g \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

If we consider this $SL_2(\mathbb{R})$ -action subject to the equivalence relation (6.12) then we will arrive to a version of the linear-fractional transformation of the operator defined in (6.2). There is a connection of the $SL_2(\mathbb{R})$ -action (6.13) to the problem (6.11) through the following intertwining relation:

Proposition 6.18. *Let λ and $x \in H$ solve the generalised eigenvalue problem (6.11) for the pair (A, B) . Then the pair $(C, D) = g \cdot (A, B)$, $g \in SL_2(\mathbb{R})$ has a solution μ and x , where*

$$\mu = g \cdot \lambda = \frac{a\lambda + b}{c\lambda + d}, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),$$

is defined by the Möbius transformation (1.1).

In other words the correspondence

$$(A, B) \mapsto \text{all generalised eigenvalues}$$

is another realisation of a covariant calculus in the sense of Defn. 4.20. The collection of all pairs $g \cdot (A, B)$, $g \in SL_2(\mathbb{R})$ is an example of *covariant pencil* of operators. This set is a $SL_2(\mathbb{R})$ -homogeneous spaces, thus it shall be within the classification of such homogeneous spaces provided in the Subsection 2.1.

Example 6.19. It is easy to demonstrate that all existing homogeneous spaces can be realised by matrix pairs.

- (i) Take the pair (O, I) where O and I are the zero and identity $n \times n$ matrices respectively. Then any transformation of this pair by a lower-triangular matrix from $SL_2(\mathbb{R})$ is equivalent to (O, I) . The respective homogeneous space is isomorphic to the real line with the Möbius transformations (1.1).
- (ii) Consider $H = \mathbb{R}^2$. Using the notations ι from Subsection 1.1 we define three realisations (elliptic, parabolic and hyperbolic) of an operator A_ι :

$$(6.14) \quad A_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for an arbitrary element h of the subgroup K , N or A the respective (in the sense of the Principle 3.5) pair $h \cdot (A_\iota, I)$ is equivalent to (A_ι, I) itself. Thus those three homogeneous spaces are isomorphic to the elliptic, parabolic and hyperbolic half-planes under respective actions of $SL_2(\mathbb{R})$. Note, that $A_\iota^2 = \iota^2 I$, that is A_ι is a model for hypercomplex units.

- (iii) Let A be a direct sum of any two different matrices out of the three A_ι from (6.14), then the fix group of the equivalence class of the pair (A, I) is the identity of $SL_2(\mathbb{R})$. Thus the corresponding homogeneous space coincides with the group itself.

Hawing homogeneous spaces generated by pairs of operators we can define respective functions on those spaces. The special attention is due the following paraphrase of the resolvent:

$$R_{(A,B)}(g) = (cA + dB)^{-1} \quad \text{where} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Obviously $R_{(A,B)}(g)$ contains the essential information about the pair (A, B) . Probably, the function $R_{(A,B)}(g)$ contains too much simultaneous information, we may restrict it to get a more detailed view. For vectors $u, v \in H$ we also consider vector and scalar-valued functions related to the generalised resolvent:

$$R_{(A,B)}^u(g) = (cA + dB)^{-1}u, \quad \text{and} \quad R_{(A,B)}^{(u,v)}(g) = \langle (cA + dB)^{-1}u, v \rangle,$$

where $(cA + dB)^{-1}u$ is understood as a solution w of the equation $u = (cA + dB)w$ if it exists and is unique, this does not require the full invertibility of $cA + dB$.

It is easy to see that the map $(A, B) \mapsto R_{(A,B)}^{(u,v)}(g)$ is a covariant calculus as well. It worth to notice that function $R_{(A,B)}$ can again fall into three EPH cases.

Example 6.20. For the three matrices A_i considered in the previous Example we denote by $R_i(g)$ the resolvent-type function of the pair (A_i, I) . Then:

$$R_i(g) = \frac{1}{c^2 + d^2} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}, \quad R_\varepsilon(g) = \frac{1}{d^2} \begin{pmatrix} d & -c \\ 0 & d \end{pmatrix}, \quad R_j(g) = \frac{1}{d^2 - c^2} \begin{pmatrix} d & -c \\ -c & d \end{pmatrix}.$$

Put $u = (1, 0) \in H$, then $R_i(g)u$ is a two-dimensional real vector valued functions with components equal to real and imaginary part of hypercomplex Cauchy kernel considered in [86].

Consider the space $L(G)$ of functions spanned by all left translations of $R_{(A,B)}(g)$. As usual, a closure in a suitable metric, say L_p , can be taken. The left action $g : f(h) \mapsto f(g^{-1}h)$ of $\mathrm{SL}_2(\mathbb{R})$ on this space is a linear representation of this group. Afterwards the representation can be decomposed into a sum of primary subrepresentations.

Example 6.21. For the matrices A_i the irreducible components are isomorphic to analytic spaces of hypercomplex functions under the fraction-linear transformations build in Subsection 3.2.

An important observation is that a decomposition into irreducible or primary components can reveal an EPH structure even in the cases hiding it on the homogeneous space level.

Example 6.22. Take the operator $A = A_i \oplus A_j$ from the Example 6.19(iii). The corresponding homogeneous space coincides with the entire $\mathrm{SL}_2(\mathbb{R})$. However if we take two vectors $u_i = (1, 0) \oplus (0, 0)$ and $u_j = (0, 0) \oplus (1, 0)$ then the respective linear spaces generated by functions $R_A(g)u_i$ and $R_A(g)u_j$ will be of elliptic and hyperbolic types respectively.

Let us briefly consider a *quadratic eigenvalue* problem: for given operators (matrices) A_0, A_1 and A_2 from $B(H)$ find a scalar λ and a vector $x \in H$ such that

$$(6.15) \quad Q(\lambda)x = 0, \quad \text{where} \quad Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0.$$

There is a connection with our study of conic sections from Subsection 2.2 which we will only hint for now. Comparing (6.15) with the equation of the cycle (2.7) we can associate the respective Fillmore–Springer–Cnops–type matrix to $Q(\lambda)$, cf. (2.8):

$$(6.16) \quad Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 \quad \longleftrightarrow \quad C_Q = \begin{pmatrix} A_1 & A_0 \\ A_2 & -A_1 \end{pmatrix}.$$

Then we can state the following analogue of Thm. 2.4 for the quadratic eigenvalues:

Proposition 6.23. *Let two quadratic matrix polynomials Q and \tilde{Q} are such that their FSC matrices (6.16) are conjugated $C_{\tilde{Q}} = gC_Qg^{-1}$ by an element $g \in \text{SL}_2(\mathbb{R})$. Then λ is a solution of the quadratic eigenvalue problem for Q and $x \in H$ if and only if $\mu = g \cdot \lambda$ is a solution of the quadratic eigenvalue problem for \tilde{Q} and x . Here $\mu = g \cdot \lambda$ is the Möbius transformation (1.1) associated to $g \in \text{SL}_2(\mathbb{R})$.*

So quadratic matrix polynomials are non-commuting analogues of the cycles and it would be exciting to extend the geometry from Section 2 to this non-commutative setting as much as possible.

Remark 6.24. It is beneficial to extend a notion of a scalar in an (generalised) eigenvalue problem to an abstract field or ring. For example, we can consider pencils of operators/matrices with polynomial coefficients. In many circumstances we may factorise the polynomial ring by an ideal generated by a collection of algebraic equations. Our work with hypercomplex units is the most elementary realisation of this setup. Indeed, the algebra of hypercomplex numbers with the hypercomplex unit ι is a realisation of the polynomial ring in a variable t factored by the single quadratic relation $t^2 + \sigma = 0$, where $\sigma = \iota^2$.

7. QUANTUM MECHANICS

Complex valued representations of the Heisenberg group (also known as Weyl or Heisenberg-Weyl group) provide a natural framework for quantum mechanics [31, 43]. This is the most fundamental example of the Kirillov orbit method, induced representations and geometrical quantisation technique [56, 57]. Following the presentation in Section 3 we will consider representations of the Heisenberg group which are induced by hypercomplex characters of its centre: complex (which correspond to the elliptic case), dual (parabolic) and double (hyperbolic).

To describe dynamics of a physical system we use a universal equation based on inner derivations (commutator) of the convolution algebra [70, 74]. The complex valued representations produce the standard framework for quantum mechanics with the Heisenberg dynamical equation [123].

The double number valued representations, with the hyperbolic unit $j^2 = 1$, is a natural source of hyperbolic quantum mechanics developed for a while [45, 46, 49, 51, 52]. The universal dynamical equation employs hyperbolic commutator in this case. This can be seen as a *Moyal bracket* based on the hyperbolic sine function. The hyperbolic observables act as operators on a Krein space with an indefinite inner product. Such spaces are employed in study of \mathcal{PT} -symmetric Hamiltonians and hyperbolic unit $j^2 = 1$ naturally appear in this setup [38].

The representations with values in dual numbers provide a convenient description of the classical mechanics. For this we do not take any sort of semiclassical limit, rather the nilpotency of the parabolic unit ($\epsilon^2 = 0$) do the task. This removes the vicious necessity to consider the Planck *constant* tending to zero. The dynamical equation takes the Hamiltonian form. We also describe classical non-commutative representations of the Heisenberg group which acts in the first jet space.

Remark 7.1. It is worth to note that our technique is different from contraction technique in the theory of Lie groups [37, 98]. Indeed a contraction of the Heisenberg group \mathbb{H}^n is the commutative Euclidean group \mathbb{R}^{2n} which does not recreate neither quantum nor classical mechanics.

The approach provides not only three different types of dynamics, it also generates the respective rules for addition of probabilities as well. For example, the quantum interference is the consequence of the same complex-valued structure, which directs the Heisenberg equation. The absence of an interference (a particle behaviour) in the classical mechanics is again the consequence the nilpotency of the parabolic unit. Double numbers creates the hyperbolic law of additions of probabilities, which was extensively investigated [49, 51]. There are still unresolved issues with positivity of the probabilistic interpretation in the hyperbolic case [45, 46].

Remark 7.2. It is commonly accepted since the Dirac's paper [25] that the striking (or even *the only*) difference between quantum and classical mechanics is non-commutativity of observables in the first case. In particular the Heisenberg commutation relations (7.5) imply the uncertainty principle, the Heisenberg equation of motion and other quantum features. However, the entire book of Feynman on QED [30] does not contain any reference to non-commutativity. Moreover, our work shows that there is a non-commutative formulation of classical mechanics. Non-commutative representations of the Heisenberg group in dual numbers implies the Poisson dynamical equation and local addition of probabilities in Section 7.6, which are completely classical.

This entirely dispels any illusive correlation between classical/quantum and commutative/non-commutative. Instead we show that quantum mechanics is fully determined by the properties of complex numbers. In Feynman's exposition [30] complex numbers are presented by a clock, rotations of its arm encode multiplications by unimodular complex numbers. Moreover, there is no a presentation of quantum mechanics, which does not employ complex phases (numbers) in one or another form. Analogous parabolic and hyperbolic phases (or characters produced by associated hypercomplex numbers, see Section 3.1) lead to classical and hypercomplex mechanics respectively.

This section clarifies foundations of quantum and classical mechanics. We recovered the existence of three non-isomorphic models of mechanics from the representation theory. They were already derived in [45, 46] from translation invariant formulation, that is from the group theory as well. It also hinted that hyperbolic counterpart is (at least theoretically) as natural as classical and quantum mechanics are. The approach provides a framework for a description of aggregate system which have say both quantum and classical components. This can be used to model quantum computers with classical terminals [80].

Remarkably, simultaneously with the work [45] group-invariant axiomatics of geometry led R.I. Pimenov [111] to description of 3^n Cayley–Klein constructions. The connection between group-invariant geometry and respective mechanics were explored in many works of N.A. Gromov, see for example [35–37]. They already highlighted the rôle of three types of hypercomplex units for the realisation of elliptic, parabolic and hyperbolic geometry and kinematic.

There is a further connection between representations of the Heisenberg group and hypercomplex numbers. The symplectomorphism of phase space are also automorphism of the Heisenberg group [31, § 1.2]. We recall that the symplectic group $\mathrm{Sp}(2)$ [31, § 1.2] is isomorphic to the group $\mathrm{SL}_2(\mathbb{R})$ [44, 96, 101] and provides linear symplectomorphisms of the two-dimensional phase space. It has three types of non-isomorphic one-dimensional continuous subgroups (2.4–2.6) with symplectic action on the phase space illustrated by Fig. 9. Hamiltonians, which produce those symplectomorphism, are of interest [118; 119; 124, § 3.8]. An analysis of those Hamiltonians from Subsection 3.3 by means of ladder operators recreates hypercomplex coefficients as well [88].

Harmonic oscillators, which we shall use as the main illustration here, are treated in most textbooks on quantum mechanics. This is efficiently done through creation/annihilation (ladder) operators, cf. § 3.3 and [13, 33]. The underlying structure is the representation theory of the Heisenberg and symplectic groups [31; 43; 96, § VI.2; 117, § 8.2]. As we will see, they are naturally connected with respective hypercomplex numbers. As a result we obtain further illustrations to the Similarity and Correspondence Principle 3.5.

We work with the simplest case of a particle with only one degree of freedom. Higher dimensions and the respective group of symplectomorphisms $\text{Sp}(2n)$ may require consideration of Clifford algebras [20, 21, 38, 60, 112].

7.1. The Heisenberg Group and Its Automorphisms.

7.1.1. *The Heisenberg group and induced representations.* Let (s, x, y) , where $s, x, y \in \mathbb{R}$, be an element of the one-dimensional Heisenberg group \mathbb{H}^1 [31, 43]. Consideration of the general case of \mathbb{H}^n will be similar, but is beyond the scope of present paper. The group law on \mathbb{H}^1 is given as follows:

$$(7.1) \quad (s, x, y) \cdot (s', x', y') = (s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y'),$$

where the non-commutativity is due to ω —the symplectic form on \mathbb{R}^{2n} , which is the central object of the classical mechanics [4, § 37]:

$$(7.2) \quad \omega(x, y; x', y') = xy' - x'y.$$

The Heisenberg group is a non-commutative Lie group with the centre

$$Z = \{(s, 0, 0) \in \mathbb{H}^1, s \in \mathbb{R}\}.$$

The left shifts

$$(7.3) \quad \Lambda(g) : f(g') \mapsto f(g^{-1}g')$$

act as a representation of \mathbb{H}^1 on a certain linear space of functions. For example, an action on $L_2(\mathbb{H}, dg)$ with respect to the Haar measure $dg = ds dx dy$ is the *left regular* representation, which is unitary.

The Lie algebra \mathfrak{h}^n of \mathbb{H}^1 is spanned by left-(right-)invariant vector fields

$$(7.4) \quad S^{l(r)} = \pm \partial_s, \quad X^{l(r)} = \pm \partial_x - \frac{1}{2}y\partial_s, \quad Y^{l(r)} = \pm \partial_y + \frac{1}{2}x\partial_s$$

on \mathbb{H}^1 with the Heisenberg commutator relation

$$(7.5) \quad [X^{l(r)}, Y^{l(r)}] = S^{l(r)}$$

and all other commutators vanishing. We will sometimes omit the superscript l for left-invariant field.

We can construct linear representations of \mathbb{H}^1 by induction [55, § 13] from a character χ of the centre Z . Here we prefer the following one, cf. § 3.2 and [55, § 13; 117, Ch. 5]. Let $F_2^X(\mathbb{H}^n)$ be the space of functions on \mathbb{H}^n having the properties:

$$(7.6) \quad f(gh) = \chi(h)f(g), \quad \text{for all } g \in \mathbb{H}^n, h \in Z$$

and

$$(7.7) \quad \int_{\mathbb{R}^{2n}} |f(0, x, y)|^2 dx dy < \infty.$$

Then $F_2^X(\mathbb{H}^n)$ is invariant under the left shifts and those shifts restricted to $F_2^X(\mathbb{H}^n)$ make a representation ρ_χ of \mathbb{H}^n induced by χ .

If the character χ is unitary, then the induced representation is unitary as well. However the representation ρ_χ is not necessarily irreducible. Indeed, left shifts are commuting with the right action of the group. Thus any subspace of null-solutions of a linear combination $aS + \sum_{j=1}^n (b_j X_j + c_j Y_j)$ of left-invariant vector

fields is left-invariant and we can restrict ρ_χ to this subspace. The left-invariant differential operators define analytic condition for functions, cf. Cor. 5.6.

Example 7.3. The function $f_0(s, x, y) = e^{ihs - h(x^2 + y^2)/4}$, where $h = 2\pi\hbar$, belongs to $F_2^X(\mathbb{H}^n)$ for the character $\chi(s) = e^{ihs}$. It is also a null solution for all the operators $X_j - iY_j$. The closed linear span of functions $f_g = \Lambda(g)f_0$ is invariant under left shifts and provide a model for Fock–Segal–Bargmann (FSB) type representation of the Heisenberg group, which will be considered below.

7.1.2. Symplectic Automorphisms of the Heisenberg Group. The group of outer automorphisms of \mathbb{H}^1 , which trivially acts on the centre of \mathbb{H}^1 , is the symplectic group $\text{Sp}(2)$. It is the group of symmetries of the symplectic form ω in (7.1) [31, Thm. 1.22; 42, p. 830]. The symplectic group is isomorphic to $\text{SL}_2(\mathbb{R})$ considered in the first half of this work. The explicit action of $\text{Sp}(2)$ on the Heisenberg group is:

$$(7.8) \quad g : h = (s, x, y) \mapsto g(h) = (s, x', y'),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2), \quad \text{and} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The Shale–Weil theorem [31, § 4.2; 42, p. 830] states that any representation ρ_h of the Heisenberg groups generates a unitary *oscillator* (or *metaplectic*) representation ρ_h^{SW} of the $\widetilde{\text{Sp}}(2)$, the two-fold cover of the symplectic group [31, Thm. 4.58].

We can consider the semidirect product $G = \mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$ with the standard group law:

$$(7.9) \quad (h, g) * (h', g') = (h * g(h'), g * g'), \quad \text{where } h, h' \in \mathbb{H}^1, \quad g, g' \in \widetilde{\text{Sp}}(2),$$

and the stars denote the respective group operations while the action $g(h')$ is defined as the composition of the projection map $\widetilde{\text{Sp}}(2) \rightarrow \text{Sp}(2)$ and the action (7.8). This group is sometimes called the *Schrödinger group* and it is known as the maximal kinematical invariance group of both the free Schrödinger equation and the quantum harmonic oscillator [104]. This group is of interest not only in quantum mechanics but also in optics [118, 119]. The Shale–Weil theorem allows us to expand any representation ρ_h of the Heisenberg group to the representation $\rho_h^2 = \rho_h \oplus \rho_h^{\text{SW}}$ of the group G .

Consider the Lie algebra \mathfrak{sp}_2 of the group $\text{Sp}(2)$. We again use the basis A, B, Z (3.12) with commutators (3.13). Vectors $Z, B - Z/2$ and B are generators of the one-parameter subgroups K, N' and A' (2.4–2.6) respectively. Furthermore we can consider the basis $\{S, X, Y, A, B, Z\}$ of the Lie algebra \mathfrak{g} of the Lie group $G = \mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$. All non-zero commutators besides those already listed in (7.5) and (3.13) are:

$$(7.10) \quad [A, X] = \frac{1}{2}X, \quad [B, X] = -\frac{1}{2}Y, \quad [Z, X] = Y;$$

$$(7.11) \quad [A, Y] = -\frac{1}{2}Y, \quad [B, Y] = -\frac{1}{2}X, \quad [Z, Y] = -X.$$

Of course, there is the derived form of the Shale–Weil representation for \mathfrak{g} . It can often be explicitly written in contrast to the Shale–Weil representation.

Example 7.4. Let ρ_h be the Schrödinger representation [31, § 1.3] of \mathbb{H}^1 in $L_2(\mathbb{R})$, that is [87, (3.5)]:

$$[\rho_\chi(s, x, y)f](q) = e^{2\pi i h(s - xy/2) + 2\pi i x q} f(q - \hbar y).$$

Thus the action of the derived representation on the Lie algebra \mathfrak{h}_1 is:

$$(7.12) \quad \rho_h(X) = 2\pi i q, \quad \rho_h(Y) = -\hbar \frac{d}{dq}, \quad \rho_h(S) = 2\pi i \hbar I.$$

Then the associated Shale–Weil representation of $\mathrm{Sp}(2)$ in $L_2(\mathbb{R})$ has the derived action, cf. [31, § 4.3; 118, (2.2)]:

(7.13)

$$\rho_h^{\mathrm{SW}}(A) = -\frac{q}{2} \frac{d}{dq} - \frac{1}{4}, \quad \rho_h^{\mathrm{SW}}(B) = -\frac{\hbar i}{8\pi} \frac{d^2}{dq^2} - \frac{\pi i q^2}{2\hbar}, \quad \rho_h^{\mathrm{SW}}(Z) = \frac{\hbar i}{4\pi} \frac{d^2}{dq^2} - \frac{\pi i q^2}{\hbar}.$$

We can verify commutators (7.5) and (3.13), (7.11) for operators (7.12–7.13). It is also obvious that in this representation the following algebraic relations hold:

(7.14)

$$\begin{aligned} \rho_h^{\mathrm{SW}}(A) &= \frac{i}{4\pi\hbar} (\rho_h(X)\rho_h(Y) - \frac{1}{2}\rho_h(S)) \\ &= \frac{i}{8\pi\hbar} (\rho_h(X)\rho_h(Y) + \rho_h(Y)\rho_h(X)), \end{aligned}$$

(7.15)

$$\rho_h^{\mathrm{SW}}(B) = \frac{i}{8\pi\hbar} (\rho_h(X)^2 - \rho_h(Y)^2),$$

(7.16)

$$\rho_h^{\mathrm{SW}}(Z) = \frac{i}{4\pi\hbar} (\rho_h(X)^2 + \rho_h(Y)^2).$$

Thus it is common in quantum optics to name \mathfrak{g} as a Lie algebra with quadratic generators, see [33, § 2.2.4].

Note that $\rho_h^{\mathrm{SW}}(Z)$ is the Hamiltonian of the harmonic oscillator (up to a factor). Then we can consider $\rho_h^{\mathrm{SW}}(B)$ as the Hamiltonian of a repulsive (hyperbolic) oscillator. The operator $\rho_h^{\mathrm{SW}}(B - Z/2) = \frac{\hbar i}{4\pi} \frac{d^2}{dq^2}$ is the parabolic analog. A graphical representation of all three transformations defined by those Hamiltonians is given in Fig. 9 and a further discussion of these Hamiltonians can be found in [124, § 3.8].

An important observation, which is often missed, is that the three linear symplectic transformations are unitary rotations in the corresponding hypercomplex algebra, cf. [85, § 3]. This means, that the symplectomorphisms generated by operators Z , $B - Z/2$, B within time t coincide with the multiplication of hypercomplex number $q + \iota p$ by $e^{\iota t}$, see Subsection 3.1 and Fig. 9, which is just another illustration of the Similarity and Correspondence Principle 3.5.

Example 7.5. There are many advantages of considering representations of the Heisenberg group on the phase space [24; 31, § 1.6; 43, § 1.7]. A convenient expression for Fock–Segal–Bargmann (FSB) representation on the phase space is, cf. § 7.3.1 and [24, (1); 74, (2.9)]:

(7.17)

$$[\rho_F(s, x, y)f](q, p) = e^{-2\pi i(\hbar s + qx + py)} f\left(q - \frac{\hbar}{2}y, p + \frac{\hbar}{2}x\right).$$

Then the derived representation of \mathfrak{h}_1 is:

(7.18)

$$\rho_F(X) = -2\pi i q + \frac{\hbar}{2} \partial_p, \quad \rho_F(Y) = -2\pi i p - \frac{\hbar}{2} \partial_q, \quad \rho_F(S) = -2\pi i \hbar I.$$

This produces the derived form of the Shale–Weil representation:

(7.19)

$$\rho_F^{\mathrm{SW}}(A) = \frac{1}{2} (q \partial_q - p \partial_p), \quad \rho_F^{\mathrm{SW}}(B) = -\frac{1}{2} (p \partial_q + q \partial_p), \quad \rho_F^{\mathrm{SW}}(Z) = p \partial_q - q \partial_p.$$

Note that this representation does not contain the parameter \hbar unlike the equivalent representation (7.13). Thus the FSB model explicitly shows the equivalence of $\rho_{\hbar_1}^{\mathrm{SW}}$ and $\rho_{\hbar_2}^{\mathrm{SW}}$ if $\hbar_1 \hbar_2 > 0$ [31, Thm. 4.57].

As we will also see below the FSB-type representations in hypercomplex numbers produce almost the same Shale–Weil representations.

7.2. p-Mechanic Formalism. Here we briefly outline a formalism [15, 63, 70, 74, 113], which allows to unify quantum and classical mechanics.

7.2.1. *Convolutions (Observables) on \mathbb{H}^n and Commutator.* Using an invariant measure $dg = ds dx dy$ on \mathbb{H}^n we can define the convolution of two functions:

$$(7.20) \quad (k_1 * k_2)(g) = \int_{\mathbb{H}^n} k_1(g_1) k_2(g_1^{-1}g) dg_1.$$

This is a non-commutative operation, which is meaningful for functions from various spaces including $L_1(\mathbb{H}^n, dg)$, the Schwartz space S and many classes of distributions, which form algebras under convolutions. Convolutions on \mathbb{H}^n are used as *observables* in p -mechanic [63, 74].

A unitary representation ρ of \mathbb{H}^n extends to $L_1(\mathbb{H}^n, dg)$ by the formula:

$$(7.21) \quad \rho(k) = \int_{\mathbb{H}^n} k(g) \rho(g) dg.$$

This is also an algebra homomorphism of convolutions to linear operators.

For a dynamics of observables we need inner *derivations* D_k of the convolution algebra $L_1(\mathbb{H}^n)$, which are given by the *commutator*:

$$(7.22) \quad \begin{aligned} D_k : f \mapsto [k, f] &= k * f - f * k \\ &= \int_{\mathbb{H}^n} k(g_1) (f(g_1^{-1}g) - f(gg_1^{-1})) dg_1, \quad f, k \in L_1(\mathbb{H}^n). \end{aligned}$$

To describe dynamics of a time-dependent observable $f(t, g)$ we use the universal equation, cf. [61, 63]:

$$(7.23) \quad S\dot{f} = [H, f],$$

where S is the left-invariant vector field (7.4) generated by the centre of \mathbb{H}^n . The presence of operator S fixes the dimensionality of both sides of the equation (7.23) if the observable H (Hamiltonian) has the dimensionality of energy [74, Rem 4.1]. If we apply a right inverse \mathcal{A} of S to both sides of the equation (7.23) we obtain the equivalent equation

$$(7.24) \quad \dot{f} = \llbracket H, f \rrbracket,$$

based on the universal bracket $\llbracket k_1, k_2 \rrbracket = k_1 * \mathcal{A}k_2 - k_2 * \mathcal{A}k_1$ [74].

Example 7.6 (Harmonic oscillator). Let $H = \frac{1}{2}(mk^2q^2 + \frac{1}{m}p^2)$ be the Hamiltonian of a one-dimensional harmonic oscillator, where k is a constant frequency and m is a constant mass. Its p -mechanisation will be the second order differential operator on \mathbb{H}^n [15, § 5.1]:

$$H = \frac{1}{2}(mk^2X^2 + \frac{1}{m}Y^2),$$

where we dropped sub-indexes of vector fields (7.4) in one dimensional setting. We can express the commutator as a difference between the left and the right action of the vector fields:

$$[H, f] = \frac{1}{2}(mk^2((X^r)^2 - (X^l)^2) + \frac{1}{m}((Y^r)^2 - (Y^l)^2))f.$$

Thus the equation (7.23) becomes [15, (5.2)]:

$$(7.25) \quad \frac{\partial}{\partial s}\dot{f} = \frac{\partial}{\partial s} \left(mk^2y \frac{\partial}{\partial x} - \frac{1}{m}x \frac{\partial}{\partial y} \right) f.$$

Of course, the derivative $\frac{\partial}{\partial s}$ can be dropped from both sides of the equation and the general solution is found to be:

$$(7.26) \quad f(t; s, x, y) = f_0 \left(s, x \cos(kt) + mky \sin(kt), -\frac{x}{mk} \sin(kt) + y \cos(kt) \right),$$

where $f_0(s, x, y)$ is the initial value of an observable on \mathbb{H}^n .

Example 7.7 (Unharmonic oscillator). We consider unharmonic oscillator with cubic potential, see [16] and references therein:

$$(7.27) \quad H = \frac{mk^2}{2}q^2 + \frac{\lambda}{6}q^3 + \frac{1}{2m}p^2.$$

Due to the absence of non-commutative products p-mechanisation is straightforward:

$$H = \frac{mk^2}{2}X^2 + \frac{\lambda}{6}X^3 + \frac{1}{m}Y^2.$$

Similarly to the harmonic case the dynamic equation, after cancellation of $\frac{\partial}{\partial s}$ on both sides, becomes:

$$(7.28) \quad \dot{f} = \left(mk^2 y \frac{\partial}{\partial x} + \frac{\lambda}{6} \left(3y \frac{\partial^2}{\partial x^2} + \frac{1}{4} y^3 \frac{\partial^2}{\partial s^2} \right) - \frac{1}{m} x \frac{\partial}{\partial y} \right) f.$$

Unfortunately, it cannot be solved analytically as easy as in the harmonic case.

7.2.2. States and Probability. Let an observable $\rho(k)$ (7.21) is defined by a kernel $k(g)$ on the Heisenberg group and its representation ρ at a Hilbert space \mathcal{H} . A state on the convolution algebra is given by a vector $v \in \mathcal{H}$. A simple calculation:

$$\begin{aligned} \langle \rho(k)v, v \rangle_{\mathcal{H}} &= \left\langle \int_{\mathbb{H}^n} k(g) \rho(g) v \, dg, v \right\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{H}^n} k(g) \langle \rho(g)v, v \rangle_{\mathcal{H}} \, dg \\ &= \int_{\mathbb{H}^n} k(g) \overline{\langle v, \rho(g)v \rangle_{\mathcal{H}}} \, dg \end{aligned}$$

can be restated as:

$$\langle \rho(k)v, v \rangle_{\mathcal{H}} = \langle k, l \rangle, \quad \text{where } l(g) = \langle v, \rho(g)v \rangle_{\mathcal{H}}.$$

Here the left-hand side contains the inner product on \mathcal{H} , while the right-hand side uses a skew-linear pairing between functions on \mathbb{H}^n based on the Haar measure integration. In other words we obtain, cf. [15, Thm. 3.11]:

Proposition 7.8. A state defined by a vector $v \in \mathcal{H}$ coincides with the linear functional given by the wavelet transform

$$(7.29) \quad l(g) = \langle v, \rho(g)v \rangle_{\mathcal{H}}$$

of v used as the mother wavelet as well.

The addition of vectors in \mathcal{H} implies the following operation on states:

$$(7.30) \quad \begin{aligned} \langle v_1 + v_2, \rho(g)(v_1 + v_2) \rangle_{\mathcal{H}} &= \langle v_1, \rho(g)v_1 \rangle_{\mathcal{H}} + \langle v_2, \rho(g)v_2 \rangle_{\mathcal{H}} \\ &\quad + \langle v_1, \rho(g)v_2 \rangle_{\mathcal{H}} + \overline{\langle v_1, \rho(g^{-1})v_2 \rangle_{\mathcal{H}}} \end{aligned}$$

The last expression can be conveniently rewritten for kernels of the functional as

$$(7.31) \quad l_{12} = l_1 + l_2 + 2A\sqrt{l_1 l_2}$$

for some real number A . This formula is behind the contextual law of addition of conditional probabilities [50] and will be illustrated below. Its physical interpretation is an interference, say, from two slits. Despite of a common belief, the mechanism of such interference can be both causal and local, see [54, 72].

7.3. Elliptic characters and Quantum Dynamics. In this subsection we consider the representation ρ_h of \mathbb{H}^n induced by the elliptic character $\chi_h(s) = e^{ihs}$ in complex numbers parametrised by $h \in \mathbb{R}$. We also use the convenient agreement $\hbar = 2\pi\hbar$ borrowed from physical literature.

7.3.1. Fock–Segal–Bargmann and Schrödinger Representations. The realisation of ρ_{\hbar} by the left shifts (7.3) on $L_2^{\hbar}(\mathbb{H}^n)$ is rarely used in quantum mechanics. Instead two unitary equivalent forms are more common: the Schrödinger and Fock–Segal–Bargmann (FSB) representations.

The FSB representation can be obtained from the orbit method of Kirillov [56]. It allows spatially separate irreducible components of the left regular representation, each of them become located on the orbit of the co-adjoint representation, see [56; 74, § 2.1] for details, we only present a brief summary here.

We identify \mathbb{H}^n and its Lie algebra \mathfrak{h}_n through the exponential map [55, § 6.4]. The dual \mathfrak{h}_n^* of \mathfrak{h}_n is presented by the Euclidean space \mathbb{R}^{2n+1} with coordinates (\hbar, q, p) . The pairing \mathfrak{h}_n^* and \mathfrak{h}_n given by

$$\langle (s, x, y), (\hbar, q, p) \rangle = \hbar s + q \cdot x + p \cdot y.$$

This pairing defines the Fourier transform $\hat{\cdot} : L_2(\mathbb{H}^n) \rightarrow L_2(\mathfrak{h}_n^*)$ given by [57, § 2.3]:

$$(7.32) \quad \hat{\phi}(F) = \int_{\mathfrak{h}^n} \phi(\exp X) e^{-2\pi i \langle X, F \rangle} dX \quad \text{where } X \in \mathfrak{h}^n, F \in \mathfrak{h}_n^*.$$

For a fixed \hbar the left regular representation (7.3) is mapped by the Fourier transform to the FSB type representation (7.17). The collection of points $(\hbar, q, p) \in \mathfrak{h}_n^*$ for a fixed \hbar is naturally identified with the *phase space* of the system.

Remark 7.9. It is possible to identify the case of $\hbar = 0$ with classical mechanics [74]. Indeed, a substitution of the zero value of \hbar into (7.17) produces the commutative representation:

$$(7.33) \quad \rho_0(s, x, y) : f(q, p) \mapsto e^{-2\pi i (qx + py)} f(q, p).$$

It can be decomposed into the direct integral of one-dimensional representations parametrised by the points (q, p) of the phase space. The classical mechanics, including the Hamilton equation, can be recovered from those representations [74]. However the condition $\hbar = 0$ (as well as the *semiclassical limit* $\hbar \rightarrow 0$) is not completely physical. Commutativity (and subsequent relative triviality) of those representation is the main reason why they are oftenly neglected. The commutativity can be outweighed by special arrangements, e.g. an antiderivative [74, (4.1)], but the procedure is not straightforward, see discussion in [1, 77, 79]. A direct approach using dual numbers will be shown below, cf. Rem. 7.21.

To recover the Schrödinger representation we use notations and technique of induced representations from § 3.2, see also [66, Ex. 4.1]. The subgroup $H = \{(s, 0, y) \mid s \in \mathbb{R}, y \in \mathbb{R}^n\} \subset \mathbb{H}^n$ defines the homogeneous space $X = G/H$, which coincides with \mathbb{R}^n as a manifold. The natural projection $\mathbf{p} : G \rightarrow X$ is $\mathbf{p}(s, x, y) = x$ and its left inverse $\mathbf{s} : X \rightarrow G$ can be as simple as $\mathbf{s}(x) = (0, x, 0)$. For the map $\mathbf{r} : G \rightarrow H$, $\mathbf{r}(s, x, y) = (s - xy/2, 0, y)$ we have the decomposition

$$(s, x, y) = \mathbf{s}(\mathbf{p}(s, x, y)) * \mathbf{r}(s, x, y) = (0, x, 0) * (s - \frac{1}{2}xy, 0, y).$$

For a character $\chi_{\hbar}(s, 0, y) = e^{i\hbar s}$ of H the lifting $\mathcal{L}_{\chi} : L_2(G/H) \rightarrow L_2^X(G)$ is as follows:

$$[\mathcal{L}_{\chi} f](s, x, y) = \chi_{\hbar}(\mathbf{r}(s, x, y)) f(\mathbf{p}(s, x, y)) = e^{i\hbar(s - xy/2)} f(x).$$

Thus the representation $\rho_{\chi}(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}$ becomes:

$$(7.34) \quad [\rho_{\chi}(s', x', y') f](x) = e^{-2\pi i \hbar (s' + xy' - x'y'/2)} f(x - x').$$

After the Fourier transform $x \mapsto q$ we get the Schrödinger representation on the *configuration space*:

$$(7.35) \quad [\rho_{\chi}(s', x', y') \hat{f}](q) = e^{-2\pi i \hbar (s' + x'y'/2) - 2\pi i x' q} \hat{f}(q + \hbar y').$$

Note that this again turns into a commutative representation (multiplication by an unimodular function) if $\hbar = 0$. To get the full set of commutative representations in this way we need to use the character $\chi_{(\hbar, p)}(s, 0, y) = e^{2\pi i(\hbar + py)}$ in the above consideration.

7.3.2. Commutator and the Heisenberg Equation. The property (7.6) of $F_2^X(\mathbb{H}^n)$ implies that the restrictions of two operators $\rho_\chi(k_1)$ and $\rho_\chi(k_2)$ to this space are equal if

$$\int_{\mathbb{R}} k_1(s, x, y) \chi(s) ds = \int_{\mathbb{R}} k_2(s, x, y) \chi(s) ds.$$

In other words, for a character $\chi(s) = e^{2\pi i \hbar s}$ the operator $\rho_\chi(k)$ depends only on

$$\hat{k}_s(\hbar, x, y) = \int_{\mathbb{R}} k(s, x, y) e^{-2\pi i \hbar s} ds,$$

which is the partial Fourier transform $s \mapsto \hbar$ of $k(s, x, y)$. The restriction to $F_2^X(\mathbb{H}^n)$ of the composition formula for convolutions is [74, (3.5)]:

$$(7.36) \quad (k' * k)_s^\wedge = \int_{\mathbb{R}^{2n}} e^{i\hbar(xy' - yx')/2} \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

Under the Schrödinger representation (7.35) the convolution (7.36) defines a rule for composition of two pseudo-differential operators (PDO) in the Weyl calculus [31, § 2.3; 43].

Consequently the representation (7.21) of commutator (7.22) depends only on its partial Fourier transform [74, (3.6)]:

$$(7.37) \quad [k', k]_s^\wedge = 2i \int_{\mathbb{R}^{2n}} \sin\left(\frac{\hbar}{2}(xy' - yx')\right) \times \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

Under the Fourier transform (7.32) this commutator is exactly the *Moyal bracket* [126] for \hat{k}' and \hat{k} on the phase space.

For observables in the space $F_2^X(\mathbb{H}^n)$ the action of S is reduced to multiplication, e.g. for $\chi(s) = e^{i\hbar s}$ the action of S is multiplication by $i\hbar$. Thus the equation (7.23) reduced to the space $F_2^X(\mathbb{H}^n)$ becomes the Heisenberg type equation [74, (4.4)]:

$$(7.38) \quad \dot{f} = \frac{1}{i\hbar} [H, f]_s^\wedge,$$

based on the above bracket (7.37). The Schrödinger representation (7.35) transforms this equation to the original Heisenberg equation.

Example 7.10. (i) Under the Fourier transform $(x, y) \mapsto (q, p)$ the p-dynamic equation (7.25) of the harmonic oscillator becomes:

$$(7.39) \quad \dot{f} = \left(mk^2 q \frac{\partial}{\partial p} - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The same transform creates its solution out of (7.26).

(ii) Since $\frac{\partial}{\partial s}$ acts on $F_2^X(\mathbb{H}^n)$ as multiplication by $i\hbar$, the quantum representation of unharmonic dynamics equation (7.28) is:

$$(7.40) \quad \dot{f} = \left(mk^2 q \frac{\partial}{\partial p} + \frac{\lambda}{6} \left(3q^2 \frac{\partial}{\partial p} - \frac{\hbar^2}{4} \frac{\partial^3}{\partial p^3} \right) - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

This is exactly the equation for the Wigner function obtained in [16, (30)].

7.3.3. Quantum Probabilities. For the elliptic character $\chi_h(s) = e^{ihs}$ we can use the Cauchy–Schwartz inequality to demonstrate that the real number A in the identity (7.31) is between -1 and 1 . Thus we can put $A = \cos \alpha$ for some angle (phase) α to get the formula for counting quantum probabilities, cf. [51, (2)]:

$$(7.41) \quad l_{12} = l_1 + l_2 + 2 \cos \alpha \sqrt{l_1 l_2}$$

Remark 7.11. It is interesting to note that the both trigonometric functions are employed in quantum mechanics: sine is in the heart of the Moyal bracket (7.37) and cosine is responsible for the addition of probabilities (7.41). In the essence the commutator and probabilities took respectively the odd and even parts of the elliptic character e^{ihs} .

Example 7.12. Take a vector $v_{(a,b)} \in L_2^h(\mathbb{H}^n)$ defined by a Gaussian with mean value (a, b) in the phase space for a harmonic oscillator of the mass m and the frequency k :

$$(7.42) \quad v_{(a,b)}(q, p) = \exp \left(-\frac{2\pi km}{\hbar} (q - a)^2 - \frac{2\pi}{\hbar km} (p - b)^2 \right).$$

A direct calculation shows:

$$\begin{aligned} \langle v_{(a,b)}, \rho_h(s, x, y) v_{(a',b')} \rangle &= \frac{4}{\hbar} \exp \left(\pi i (2s\hbar + x(a + a') + y(b + b')) \right) \\ &\quad - \frac{\pi}{2\hbar km} ((\hbar x + b - b')^2 + (b - b')^2) - \frac{\pi km}{2\hbar} ((\hbar y + a' - a)^2 + (a' - a)^2) \\ &= \frac{4}{\hbar} \exp \left(\pi i (2s\hbar + x(a + a') + y(b + b')) \right) \\ &\quad - \frac{\pi}{\hbar km} \left((b - b' + \frac{\hbar x}{2})^2 + (\frac{\hbar x}{2})^2 \right) - \frac{\pi km}{\hbar} \left((a - a' - \frac{\hbar y}{2})^2 + (\frac{\hbar y}{2})^2 \right) \end{aligned}$$

Thus the kernel $l_{(a,b)} = \langle v_{(a,b)}, \rho_h(s, x, y) v_{(a,b)} \rangle$ (7.29) for a state $v_{(a,b)}$ is:

$$(7.43) \quad l_{(a,b)} = \frac{4}{\hbar} \exp \left(2\pi i (s\hbar + xa + yb) - \frac{\pi\hbar}{2km} x^2 - \frac{\pi km\hbar}{2\hbar} y^2 \right)$$

An observable registering a particle at a point $q = c$ of the configuration space is $\delta(q - c)$. On the Heisenberg group this observable is given by the kernel:

$$(7.44) \quad X_c(s, x, y) = e^{2\pi i (s\hbar + xc)} \delta(y).$$

The measurement of X_c on the state (7.42) (through the kernel (7.43)) predictably is:

$$\langle X_c, l_{(a,b)} \rangle = \sqrt{\frac{2km}{\hbar}} \exp \left(-\frac{2\pi km}{\hbar} (c - a)^2 \right).$$

Example 7.13. Now take two states $v_{(0,b)}$ and $v_{(0,-b)}$, where for the simplicity we assume the mean values of coordinates vanish in the both cases. Then the corresponding kernel (7.30) has the interference terms:

$$\begin{aligned} l_i &= \langle v_{(0,b)}, \rho_h(s, x, y) v_{(0,-b)} \rangle \\ &= \frac{4}{\hbar} \exp \left(2\pi i s\hbar - \frac{\pi}{2\hbar km} ((\hbar x + 2b)^2 + 4b^2) - \frac{\pi\hbar km}{2} y^2 \right). \end{aligned}$$

The measurement of X_c (7.44) on this term contains the oscillating part:

$$\langle X_c, l_i \rangle = \sqrt{\frac{2km}{\hbar}} \exp \left(-\frac{2\pi km}{\hbar} c^2 - \frac{2\pi}{km\hbar} b^2 + \frac{4\pi i}{\hbar} cb \right)$$

Therefore on the kernel l corresponding to the state $v_{(0,b)} + v_{(0,-b)}$ the measurement is

$$\langle X_c, l \rangle = 2\sqrt{\frac{2km}{\hbar}} \exp\left(-\frac{2\pi km}{\hbar} c^2\right) \left(1 + \exp\left(-\frac{2\pi}{km\hbar} b^2\right) \cos\left(\frac{4\pi}{\hbar} cb\right)\right).$$

The presence of the cosine term in the last expression can generate an interference

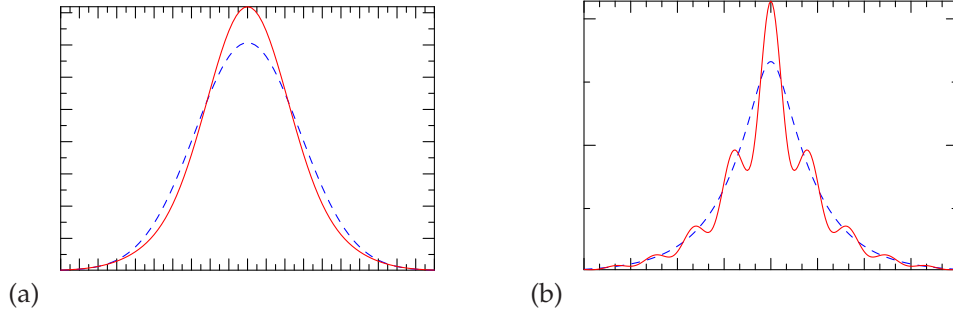


FIGURE 13. Quantum probabilities: the blue (dashed) graph shows the addition of probabilities without interaction, the red (solid) graph present the quantum interference. Left picture shows the Gaussian state (7.42), the right—the rational state (7.45)

picture. In practise it does not happen for the minimal uncertainty state (7.42) which we are using here: it rapidly vanishes outside of the neighbourhood of zero, where oscillations of the cosine occurs, see Fig. 13(a).

Example 7.14. To see a traditional interference pattern one can use a state which is far from the minimal uncertainty. For example, we can consider the state:

$$(7.45) \quad u_{(a,b)}(q, p) = \frac{\hbar^2}{((q-a)^2 + \hbar/km)((p-b)^2 + \hbar km)}.$$

To evaluate the observable X_c (7.44) on the state $l(g) = \langle u_1, \rho_h(g) u_2 \rangle$ (7.29) we use the following formula:

$$\langle X_c, l \rangle = \frac{2}{\hbar} \int_{\mathbb{R}^n} \hat{u}_1(q, 2(q-c)/\hbar) \overline{\hat{u}_2(q, 2(q-c)/\hbar)} dq,$$

where $\hat{u}_i(q, x)$ denotes the partial Fourier transform $p \mapsto x$ of $u_i(q, p)$. The formula is obtained by swapping order of integrations. The numerical evaluation of the state obtained by the addition $u_{(0,b)} + u_{(0,-b)}$ is plotted on Fig. 13(b), the red curve shows the canonical interference pattern.

7.4. Ladder Operators and Harmonic Oscillator. Let ρ be a representation of the Schrödinger group $G = \mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$ (7.9) in a space V . Consider the derived representation of the Lie algebra \mathfrak{g} [96, § VI.1] and denote $\tilde{X} = \rho(X)$ for $X \in \mathfrak{g}$. To see the structure of the representation ρ we can decompose the space V into eigenspaces of the operator \tilde{X} for some $X \in \mathfrak{g}$. The canonical example is the Taylor series in complex analysis.

We are going to consider three cases corresponding to three non-isomorphic subgroups (2.4–2.6) of $\text{Sp}(2)$ starting from the compact case. Let $H = Z$ be a generator of the compact subgroup K . Corresponding symplectomorphisms (7.8) of the phase space are given by orthogonal rotations with matrices $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. The Shale–Weil representation (7.13) coincides with the Hamiltonian of the harmonic oscillator in Schrödinger representation.

Since $\widetilde{\text{Sp}}(2)$ is a two-fold cover the corresponding eigenspaces of a compact group $\tilde{Z}v_k = ikv_k$ are parametrised by a half-integer $k \in \mathbb{Z}/2$. Explicitly for a half-integer k eigenvectors are:

$$(7.46) \quad v_k(q) = H_{k+\frac{1}{2}} \left(\sqrt{\frac{2\pi}{\hbar}} q \right) e^{-\frac{\pi}{\hbar} q^2},$$

where H_k is the *Hermite polynomial* [28, 8.2(9); 31, § 1.7].

From the point of view of quantum mechanics as well as the representation theory it is beneficial to introduce the ladder operators L^\pm (3.14), known also as *creation/annihilation* in quantum mechanics [13; 31, p. 49]. There are two ways to search for ladder operators: in (complexified) Lie algebras \mathfrak{h}_1 and \mathfrak{sp}_2 . The later coincides with our consideration in Section 3.3 in the essence.

7.4.1. Ladder Operators from the Heisenberg Group. Assuming $L^+ = a\tilde{X} + b\tilde{Y}$ we obtain from the relations (7.10–7.11) and (3.14) the linear equations with unknown a and b :

$$a = \lambda_+ b, \quad -b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 1 = 0$, and the raising/lowering operators are $L^\pm = \tilde{X} \mp i\tilde{Y}$.

Remark 7.15. Here we have an interesting asymmetric response: due to the structure of the semidirect product $\mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$ it is the symplectic group which acts on \mathbb{H}^1 , not vice versa. However the Heisenberg group has a weak action in the opposite direction: it shifts eigenfunctions of $\text{Sp}(2)$.

In the Schrödinger representation (7.12) the ladder operators are

$$(7.47) \quad \rho_{\hbar}(L^\pm) = 2\pi i q \pm i\hbar \frac{d}{dq}.$$

The standard treatment of the harmonic oscillator in quantum mechanics, which can be found in many textbooks, e.g. [31, § 1.7; 33, § 2.2.3], is as follows. The vector $v_{-1/2}(q) = e^{-\pi q^2/\hbar}$ is an eigenvector of \tilde{Z} with the eigenvalue $-\frac{i}{2}$. In addition $v_{-1/2}$ is annihilated by L^+ . Thus the chain (3.16) terminates to the right and the complete set of eigenvectors of the harmonic oscillator Hamiltonian is presented by $(L^-)^k v_{-1/2}$ with $k = 0, 1, 2, \dots$

We can make a wavelet transform generated by the Heisenberg group with the mother wavelet $v_{-1/2}$, and the image will be the Fock–Segal–Bargmann (FSB) space [31, § 1.6; 43]. Since $v_{-1/2}$ is the null solution of $L^+ = \tilde{X} - i\tilde{Y}$, then by Cor. 5.6 the image of the wavelet transform will be null-solutions of the corresponding linear combination of the Lie derivatives (7.4):

$$(7.48) \quad D = \overline{X^r - iY^r} = (\partial_x + i\partial_y) - \pi\hbar(x - iy),$$

which turns out to be the Cauchy–Riemann equation on a weighted FSB-type space.

7.4.2. Symplectic Ladder Operators. We can also look for ladder operators within the Lie algebra \mathfrak{sp}_2 , see § 3.3.1 and [85, § 8]. Assuming $L_2^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ from the relations (3.13) and defining condition (3.14) we obtain the linear equations with unknown a , b and c :

$$c = 0, \quad 2a = \lambda_+ b, \quad -2b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 4 = 0$, and the raising/lowering operators are $L_2^\pm = \pm i\tilde{A} + \tilde{B}$. In the Shale–Weil representation (7.13) they turn out

to be:

$$(7.49) \quad L_2^\pm = \pm i \left(\frac{q}{2} \frac{d}{dq} + \frac{1}{4} \right) - \frac{\hbar i}{8\pi} \frac{d^2}{dq^2} - \frac{\pi i q^2}{2\hbar} = -\frac{i}{8\pi\hbar} \left(\mp 2\pi q + \hbar \frac{d}{dq} \right)^2.$$

Since this time $\lambda_+ = 2i$ the ladder operators L_2^\pm produce a shift on the diagram (3.16) twice bigger than the operators L^\pm from the Heisenberg group. After all, this is not surprising since from the explicit representations (7.47) and (7.49) we get:

$$L_2^\pm = -\frac{i}{8\pi\hbar} (L^\pm)^2.$$

7.5. Hyperbolic Quantum Mechanics. Now we turn to double numbers also known as hyperbolic, split-complex, etc. numbers [53; 121; 125, App. C]. They form a two dimensional algebra \mathbb{O} spanned by 1 and j with the property $j^2 = 1$. There are zero divisors:

$$j_\pm = \frac{1}{\sqrt{2}}(1 \pm j), \quad \text{such that} \quad j_+ j_- = 0 \quad \text{and} \quad j_\pm^2 = j_\pm.$$

Thus double numbers algebraically isomorphic to two copies of \mathbb{R} spanned by j_\pm . Being algebraically dull double numbers are nevertheless interesting as a homogeneous space [82, 85] and they are relevant in physics [49, 121, 122]. The combination of p-mechanical approach with hyperbolic quantum mechanics was already discussed in [15, § 6].

For the hyperbolic character $\chi_{jh}(s) = e^{jhs} = \cosh hs + j \sinh hs$ of \mathbb{R} one can define the hyperbolic Fourier-type transform:

$$\hat{k}(q) = \int_{\mathbb{R}} k(x) e^{-jqx} dx.$$

It can be understood in the sense of distributions on the space dual to the set of analytic functions [52, § 3]. Hyperbolic Fourier transform intertwines the derivative $\frac{d}{dx}$ and multiplication by jq [52, Prop. 1].

Example 7.16. For the Gaussian the hyperbolic Fourier transform is the ordinary function (note the sign difference!):

$$\int_{\mathbb{R}} e^{-x^2/2} e^{-jqx} dx = \sqrt{2\pi} e^{q^2/2}.$$

However the opposite identity:

$$\int_{\mathbb{R}} e^{x^2/2} e^{-jqx} dx = \sqrt{2\pi} e^{-q^2/2}$$

is true only in a suitable distributional sense. To this end we may note that $e^{x^2/2}$ and $e^{-q^2/2}$ are null solutions to the differential operators $\frac{d}{dx} - x$ and $\frac{d}{dq} + q$ respectively, which are intertwined (up to the factor j) by the hyperbolic Fourier transform. The above differential operators $\frac{d}{dx} - x$ and $\frac{d}{dq} + q$ are images of the ladder operators (7.47) in the Lie algebra of the Heisenberg group. They are intertwining by the Fourier transform, since this is an automorphism of the Heisenberg group [42].

An elegant theory of hyperbolic Fourier transform may be achieved by a suitable adaptation of [42], which uses representation theory of the Heisenberg group.

7.5.1. *Hyperbolic Representations of the Heisenberg Group.* Consider the space $F_h^j(\mathbb{H}^n)$ of \mathbb{O} -valued functions on \mathbb{H}^n with the property:

$$(7.50) \quad f(s + s', h, y) = e^{jhs'} f(s, x, y), \quad \text{for all } (s, x, y) \in \mathbb{H}^n, s' \in \mathbb{R},$$

and the square integrability condition (7.7). Then the hyperbolic representation is obtained by the restriction of the left shifts to $F_h^j(\mathbb{H}^n)$. To obtain an equivalent representation on the phase space we take \mathbb{O} -valued functional of the Lie algebra \mathfrak{h}_n :

$$(7.51) \quad \chi_{(h,q,p)}^j(s, x, y) = e^{j(hs + qx + py)} = \cosh(hs + qx + py) + j \sinh(hs + qx + py).$$

The hyperbolic Fock–Segal–Bargmann type representation is intertwined with the left group action by means of the Fourier transform (7.32) with the hyperbolic functional (7.51). Explicitly this representation is:

$$(7.52) \quad \rho_h(s, x, y) : f(q, p) \mapsto e^{-j(hs + qx + py)} f\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right).$$

For a hyperbolic Schrödinger type representation we again use the scheme described in § 3.2. Similarly to the elliptic case one obtains the formula, resembling (7.34):

$$(7.53) \quad [\rho_X^j(s', x', y')f](x) = e^{-jh(s' + xy' - x'y'/2)} f(x - x').$$

Application of the hyperbolic Fourier transform produces a Schrödinger type representation on the configuration space, cf. (7.35):

$$[\rho_X^j(s', x', y')\hat{f}](q) = e^{-jh(s' + x'y'/2) - jx'q} \hat{f}(q + hy').$$

The extension of this representation to kernels according to (7.21) generates hyperbolic pseudodifferential operators introduced in [52, (3.4)].

7.5.2. *Hyperbolic Dynamics.* Similarly to the elliptic (quantum) case we consider a convolution of two kernels on \mathbb{H}^n restricted to $F_h^j(\mathbb{H}^n)$. The composition law becomes, cf. (7.36):

$$(7.54) \quad (k' * k)_s^\wedge = \int_{\mathbb{R}^{2n}} e^{jh(xy' - yx')} \hat{k}'_s(h, x', y') \hat{k}_s(h, x - x', y - y') dx' dy'.$$

This is close to the calculus of hyperbolic PDO obtained in [52, Thm. 2]. Respectively for the commutator of two convolutions we get, cf. (7.37):

$$(7.55) \quad [k', k]_s^\wedge = \int_{\mathbb{R}^{2n}} \sinh(h(xy' - yx')) \hat{k}'_s(h, x', y') \hat{k}_s(h, x - x', y - y') dx' dy'.$$

This is the hyperbolic version of the Moyal bracket, cf. [52, p. 849], which generates the corresponding image of the dynamic equation (7.23).

Example 7.17. (i) For a quadratic Hamiltonian, e.g. harmonic oscillator from Example 7.6, the hyperbolic equation and respective dynamics is identical to quantum considered before.

(ii) Since $\frac{\partial}{\partial s}$ acts on $F_2^j(\mathbb{H}^n)$ as multiplication by $j\hbar$ and $j^2 = 1$, the hyperbolic image of the unharmonic equation (7.28) becomes:

$$\dot{f} = \left(mk^2 q \frac{\partial}{\partial p} + \frac{\lambda}{6} \left(3q^2 \frac{\partial}{\partial p} + \frac{\hbar^2}{4} \frac{\partial^3}{\partial p^3} \right) - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The difference with quantum mechanical equation (7.40) is in the sign of the cubic derivative.

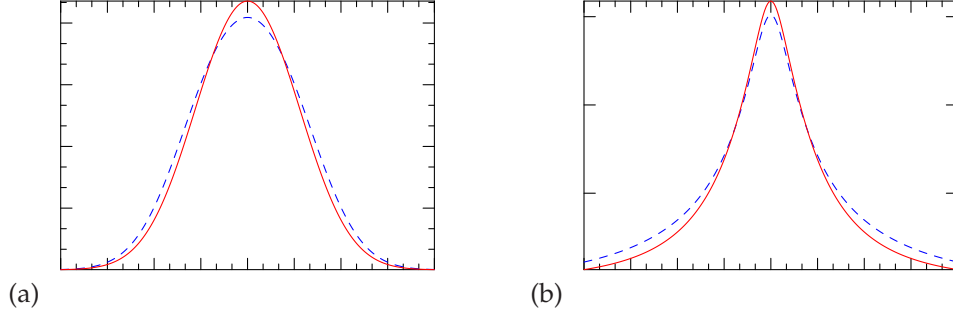


FIGURE 14. Hyperbolic probabilities: the blue (dashed) graph shows the addition of probabilities without interaction, the red (solid) graph present the quantum interference. Left picture shows the Gaussian state (7.42), with the same distribution as in quantum mechanics, cf. Fig. 13(a). The right picture shows the rational state (7.45), note the absence of interference oscillations in comparison with the quantum state on Fig. 13(b).

7.5.3. *Hyperbolic Probabilities.* To calculate probability distribution generated by a hyperbolic state we are using the general procedure from Section 7.2.2. The main differences with the quantum case are as follows:

- (i) The real number A in the expression (7.31) for the addition of probabilities is bigger than 1 in absolute value. Thus it can be associated with the hyperbolic cosine $\cosh \alpha$, cf. Rem. 7.11, for certain phase $\alpha \in \mathbb{R}$ [52].
- (ii) The nature of hyperbolic interference on two slits is affected by the fact that e^{jhs} is not periodic and the hyperbolic exponent e^{jt} and cosine $\cosh t$ do not oscillate. It is worth to notice that for Gaussian states the hyperbolic interference is exactly the same as quantum one, cf. Figs. 13(a) and 14(a). This is similar to coincidence of quantum and hyperbolic dynamics of harmonic oscillator.

The contrast between two types of interference is prominent for the rational state (7.45), which is far from the minimal uncertainty, see the different patterns on Figs. 13(b) and 14(b).

7.5.4. *Ladder Operators for the Hyperbolic Subgroup.* Consider the case of the Hamiltonian $H = 2B$, which is a repulsive (hyperbolic) harmonic oscillator [124, § 3.8]. The corresponding one-dimensional subgroup of symplectomorphisms produces hyperbolic rotations of the phase space, see Fig. 9. The eigenvectors v_μ of the operator

$$\rho_h^{SW}(2B)v_v = -i \left(\frac{\hbar}{4\pi} \frac{d^2}{dq^2} + \frac{\pi q^2}{\hbar} \right) v_v = i v v_v,$$

are *Weber–Hermite* (or *parabolic cylinder*) functions $v_v = D_{v-\frac{1}{2}}(\pm 2e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{\hbar}}q)$, see [28, § 8.2; 114] for fundamentals of Weber–Hermite functions and [118] for further illustrations and applications in optics.

The corresponding one-parameter group is not compact and the eigenvalues of the operator $2\tilde{B}$ are not restricted by any integrality condition, but the raising/lowering operators are still important [44, § II.1; 101, § 1.1]. We again seek solutions in two subalgebras \mathfrak{h}_1 and \mathfrak{sp}_2 separately. However the additional options will be provided by a choice of the number system: either complex or double.

Example 7.18 (Complex Ladder Operators). Assuming $L_h^+ = a\tilde{X} + b\tilde{Y}$ from the commutators (7.10–7.11) we obtain the linear equations:

$$(7.56) \quad -a = \lambda_+ b, \quad -b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 - 1 = 0$. Taking the real roots $\lambda = \pm 1$ we obtain that the raising/lowering operators are $L_h^\pm = \tilde{X} \mp \tilde{Y}$. In the Schrödinger representation (7.12) the ladder operators are

$$(7.57) \quad L_h^\pm = 2\pi i q \pm \hbar \frac{d}{dq}.$$

The null solutions $v_{\pm\frac{1}{2}}(q) = e^{\pm\frac{\pi i}{\hbar} q^2}$ to operators $\rho_h(L^\pm)$ are also eigenvectors of the Hamiltonian $\rho_h^{SW}(2B)$ with the eigenvalue $\pm\frac{1}{2}$. However the important distinction from the elliptic case is, that they are not square-integrable on the real line anymore.

We can also look for ladder operators within the \mathfrak{sp}_2 , that is in the form $L_{2h}^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ for the commutator $[2\tilde{B}, L_h^+] = \lambda L_h^+$, see § 3.3.2. Within complex numbers we get only the values $\lambda = \pm 2$ with the ladder operators $L_{2h}^\pm = \pm 2\tilde{A} + \tilde{Z}/2$, see [44, § II.1; 101, § 1.1]. Each indecomposable \mathfrak{h}_1 - or \mathfrak{sp}_2 -module is formed by a one-dimensional chain of eigenvalues with a transitive action of ladder operators L_h^\pm or L_{2h}^\pm respectively. And we again have a quadratic relation between the ladder operators:

$$L_{2h}^\pm = \frac{i}{4\pi\hbar} (L_h^\pm)^2.$$

7.5.5. Double Ladder Operators. There are extra possibilities in the context of hyperbolic quantum mechanics [49, 51, 52]. Here we use the representation of \mathbb{H}^1 induced by a hyperbolic character $e^{jht} = \cosh(ht) + j \sinh(ht)$, see [87, (4.5)], and obtain the hyperbolic representation of \mathbb{H}^1 , cf. (7.35):

$$(7.58) \quad [\rho_h^j(s', x', y')\hat{f}](q) = e^{jh(s' - x'y'/2) + jx'q} \hat{f}(q - hy').$$

The corresponding derived representation is

$$(7.59) \quad \rho_h^j(X) = jq, \quad \rho_h^j(Y) = -\hbar \frac{d}{dq}, \quad \rho_h^j(S) = j\hbar I.$$

Then the associated Shale–Weil derived representation of \mathfrak{sp}_2 in the Schwartz space $S(\mathbb{R})$ is, cf. (7.13):

$$(7.60) \quad \rho_h^{SW}(A) = -\frac{q}{2} \frac{d}{dq} - \frac{1}{4}, \quad \rho_h^{SW}(B) = \frac{j\hbar}{4} \frac{d^2}{dq^2} - \frac{jq^2}{4\hbar}, \quad \rho_h^{SW}(Z) = -\frac{j\hbar}{2} \frac{d^2}{dq^2} - \frac{jq^2}{2\hbar}.$$

Note that $\rho_h^{SW}(B)$ now generates a usual harmonic oscillator, not the repulsive one like $\rho_h^{SW}(B)$ in (7.13). However the expressions in the quadratic algebra are still the same (up to a factor), cf. (7.14–7.16):

$$(7.61) \quad \begin{aligned} \rho_h^{SW}(A) &= -\frac{j}{2\hbar} (\rho_h^j(X)\rho_h^j(Y) - \frac{1}{2}\rho_h^j(S)) \\ &= -\frac{j}{4\hbar} (\rho_h^j(X)\rho_h^j(Y) + \rho_h^j(Y)\rho_h^j(X)), \end{aligned}$$

$$(7.62) \quad \rho_h^{SW}(B) = \frac{j}{4\hbar} (\rho_h^j(X)^2 - \rho_h^j(Y)^2),$$

$$(7.63) \quad \rho_h^{SW}(Z) = -\frac{j}{2\hbar} (\rho_h^j(X)^2 + \rho_h^j(Y)^2).$$

This is due to the Principle 3.5 of similarity and correspondence: we can swap operators Z and B with simultaneous replacement of hypercomplex units i and j .

The eigenspace of the operator $2\rho_h^{\text{SW}}(B)$ with an eigenvalue $j\nu$ are spanned by the Weber–Hermite functions $D_{-\nu-\frac{1}{2}}\left(\pm\sqrt{\frac{2}{h}}x\right)$, see [28, § 8.2]. Functions D_ν are generalisations of the Hermit functions (7.46).

The compatibility condition for a ladder operator within the Lie algebra \mathfrak{h}_1 will be (7.56) as before, since it depends only on the commutators (7.10–7.11). Thus we still have the set of ladder operators corresponding to values $\lambda = \pm 1$:

$$L_h^\pm = \tilde{X} \mp \tilde{Y} = jq \pm h \frac{d}{dq}.$$

Admitting double numbers we have an extra way to satisfy $\lambda^2 = 1$ in (7.56) with values $\lambda = \pm j$. Then there is an additional pair of hyperbolic ladder operators, which are identical (up to factors) to (7.47):

$$L_j^\pm = \tilde{X} \mp j\tilde{Y} = jq \pm jh \frac{d}{dq}.$$

Pairs L_h^\pm and L_j^\pm shift eigenvectors in the “orthogonal” directions changing their eigenvalues by ± 1 and $\pm j$. Therefore an indecomposable \mathfrak{sp}_2 -module can be parametrised by a two-dimensional lattice of eigenvalues in double numbers, see Fig. 10.

The following functions

$$\begin{aligned} v_{\frac{1}{2}}^{\pm h}(q) &= e^{\mp jq^2/(2h)} = \cosh \frac{q^2}{2h} \mp j \sinh \frac{q^2}{2h}, \\ v_{\frac{1}{2}}^{\pm j}(q) &= e^{\mp q^2/(2h)} \end{aligned}$$

are null solutions to the operators L_h^\pm and L_j^\pm respectively. They are also eigenvectors of $2\rho_h^{\text{SW}}(B)$ with eigenvalues $\mp \frac{1}{2}$ and $\mp \frac{1}{2}$ respectively. If these functions are used as mother wavelets for the wavelet transforms generated by the Heisenberg group, then the image space will consist of the null-solutions of the following differential operators, see Cor. 5.6:

$$D_h = \overline{X^r - Y^r} = (\partial_x - \partial_y) + \frac{h}{2}(x + y), \quad D_j = \overline{X^r - jY^r} = (\partial_x + j\partial_y) - \frac{h}{2}(x - jy),$$

for $v_{\frac{1}{2}}^{\pm h}$ and $v_{\frac{1}{2}}^{\pm j}$ respectively. This is again in line with the classical result (7.48). However annihilation of the eigenvector by a ladder operator does not mean that the part of the 2D-lattice becomes void since it can be reached via alternative routes on this lattice. Instead of multiplication by a zero, as it happens in the elliptic case, a half-plane of eigenvalues will be multiplied by the divisors of zero $1 \pm j$.

We can also search ladder operators within the algebra \mathfrak{sp}_2 and admitting double numbers we will again find two sets of them, cf. § 3.3.2:

$$\begin{aligned} L_{2h}^\pm &= \pm \tilde{A} + \tilde{Z}/2 = \mp \frac{q}{2} \frac{d}{dq} \mp \frac{1}{4} - \frac{jh}{4} \frac{d^2}{dq^2} - \frac{jq^2}{4h} = -\frac{j}{4h} (L_h^\pm)^2, \\ L_{2j}^\pm &= \pm j\tilde{A} + \tilde{Z}/2 = \mp \frac{jq}{2} \frac{d}{dq} \mp \frac{j}{4} - \frac{jh}{4} \frac{d^2}{dq^2} - \frac{jq^2}{4h} = -\frac{j}{4h} (L_j^\pm)^2. \end{aligned}$$

Again the operators L_{2h}^\pm and L_{2j}^\pm produce double shifts in the orthogonal directions on the same two-dimensional lattice in Fig. 10.

7.6. Parabolic (Classical) Representations on the Phase Space. After the previous two cases it is natural to link classical mechanics with dual numbers generated by the parabolic unit $\epsilon^2 = 0$. Connection of the parabolic unit ϵ with the Galilean group of symmetries of classical mechanics is around for a while [125, App. C].

However the nilpotency of the parabolic unit ϵ make it difficult if we will work with dual number valued functions only. To overcome this issue we consider a

commutative real algebra \mathfrak{C} spanned by 1, i , ε and $i\varepsilon$ with identities $i^2 = -1$ and $\varepsilon^2 = 0$. A seminorm on \mathfrak{C} is defined as follows:

$$|a + bi + c\varepsilon + di\varepsilon|^2 = a^2 + b^2.$$

7.6.1. Classical Non-Commutative Representations. We wish to build a representation of the Heisenberg group which will be a classical analog of the Fock–Segal–Bargmann representation (7.17). To this end we introduce the space $F_h^\varepsilon(\mathbb{H}^n)$ of \mathfrak{C} -valued functions on \mathbb{H}^n with the property:

$$(7.64) \quad f(s + s', h, y) = e^{\varepsilon h s'} f(s, x, y), \quad \text{for all } (s, x, y) \in \mathbb{H}^n, s' \in \mathbb{R},$$

and the square integrability condition (7.7). It is invariant under the left shifts and we restrict the left group action to $F_h^\varepsilon(\mathbb{H}^n)$.

There is an unimodular \mathfrak{C} -valued function on the Heisenberg group parametrised by a point $(h, q, p) \in \mathbb{R}^{2n+1}$:

$$E_{(h,q,p)}(s, x, y) = e^{2\pi(\varepsilon s h + i x q + i y p)} = e^{2\pi i(x q + y p)}(1 + \varepsilon s h).$$

This function, if used instead of the ordinary exponent, produces a modification \mathcal{F}_ε of the Fourier transform (7.32). The transform intertwines the left regular representation with the following action on \mathfrak{C} -valued functions on the phase space:

$$(7.65) \quad \rho_h^\varepsilon(s, x, y) : f(q, p) \mapsto e^{-2\pi i(x q + y p)} (f(q, p) + \varepsilon h(s f(q, p) + \frac{y}{2\pi i} f'_q(q, p) - \frac{x}{2\pi i} f'_p(q, p))).$$

Remark 7.19. Comparing the traditional infinite-dimensional (7.17) and one-dimensional (7.33) representations of \mathbb{H}^n we can note that the properties of the representation (7.65) are a non-trivial mixture of the former:

- (i) The action (7.65) is non-commutative, similarly to the quantum representation (7.17) and unlike the classical one (7.33). This non-commutativity will produce the Hamilton equations below in a way very similar to Heisenberg equation, see Rem. 7.21.
- (ii) The representation (7.65) does not change the support of a function f on the phase space, similarly to the classical representation (7.33) and unlike the quantum one (7.17). Such a localised action will be responsible later for an absence of an interference in classical probabilities.
- (iii) The parabolic representation (7.65) can not be derived from either the elliptic (7.17) or hyperbolic (7.52) by the plain substitution $h = 0$.

We may also write a classical Schrödinger type representation. According to § 3.2 we get a representation formally very similar to the elliptic (7.34) and hyperbolic versions (7.53):

$$(7.66) \quad \begin{aligned} [\rho_\chi^\varepsilon(s', x', y') f](x) &= e^{-\varepsilon h(s' + x y' - x' y' / 2)} f(x - x') \\ &= (1 - \varepsilon h(s' + x y' - \tfrac{1}{2} x' y')) f(x - x'). \end{aligned}$$

However due to nilpotency of ε the (complex) Fourier transform $x \mapsto q$ produces a different formula for parabolic Schrödinger type representation in the configuration space, cf. (7.35) and (7.58):

$$[\rho_\chi^\varepsilon(s', x', y') \hat{f}](q) = e^{2\pi i x' q} \left((1 - \varepsilon h(s' - \tfrac{1}{2} x' y')) \hat{f}(q) + \frac{\varepsilon h y'}{2\pi i} \hat{f}'(q) \right).$$

This representation shares all properties mentioned in Rem. 7.19 as well.

7.6.2. Hamilton Equation. The identity $e^{\varepsilon t} - e^{-\varepsilon t} = 2\varepsilon t$ can be interpreted as a parabolic version of the sine function, while the parabolic cosine is identically equal to one, cf. § 3.1 and [40, 81]. From this we obtain the parabolic version of the commutator (7.37):

$$\begin{aligned} [k', k]_{\hat{s}}(\varepsilon h, x, y) &= \varepsilon h \int_{\mathbb{R}^{2n}} (xy' - yx') \\ &\quad \times \hat{k}'_s(\varepsilon h, x', y') \hat{k}_s(\varepsilon h, x - x', y - y') dx' dy', \end{aligned}$$

for the partial parabolic Fourier-type transform \hat{k}_s of the kernels. Thus the parabolic representation of the dynamical equation (7.23) becomes:

$$(7.67) \quad \varepsilon h \frac{d\hat{f}_s}{dt}(\varepsilon h, x, y; t) = \varepsilon h \int_{\mathbb{R}^{2n}} (xy' - yx') \hat{H}_s(\varepsilon h, x', y') \hat{f}_s(\varepsilon h, x - x', y - y'; t) dx' dy',$$

Although there is no possibility to divide by ε (since it is a zero divisor) we can obviously eliminate εh from the both sides if the rest of the expressions are real. Moreover this can be done “in advance” through a kind of the antiderivative operator considered in [74, (4.1)]. This will prevent “imaginary parts” of the remaining expressions (which contain the factor ε) from vanishing.

Remark 7.20. It is noteworthy that the Planck constant completely disappeared from the dynamical equation. Thus the only prediction about it following from our construction is $\hbar \neq 0$, which was confirmed by experiments, of course.

Using the duality between the Lie algebra of \mathbb{H}^n and the phase space we can find an adjoint equation for observables on the phase space. To this end we apply the usual Fourier transform $(x, y) \mapsto (q, p)$. It turn to be the Hamilton equation [74, (4.7)]. However the transition to the phase space is more a custom rather than a necessity and in many cases we can efficiently work on the Heisenberg group itself.

Remark 7.21. It is noteworthy, that the non-commutative representation (7.65) allows to obtain the Hamilton equation directly from the commutator $[\rho_h^\varepsilon(k_1), \rho_h^\varepsilon(k_2)]$. Indeed its straightforward evaluation will produce exactly the above expression. On the contrast such a commutator for the commutative representation (7.33) is zero and to obtain the Hamilton equation we have to work with an additional tools, e.g. an anti-derivative [74, (4.1)].

Example 7.22. (i) For the harmonic oscillator in Example 7.6 the equation (7.67) again reduces to the form (7.25) with the solution given by (7.26). The adjoint equation of the harmonic oscillator on the phase space is not different from the quantum written in Example 7.10(i). This is true for any Hamiltonian of at most quadratic order.
(ii) For non-quadratic Hamiltonians classical and quantum dynamics are different, of course. For example, the cubic term of ∂_s in the equation (7.28) will generate the factor $\varepsilon^3 = 0$ and thus vanish. Thus the equation (7.67) of the unharmonic oscillator on \mathbb{H}^n becomes:

$$\dot{f} = \left(mk^2 y \frac{\partial}{\partial x} + \frac{\lambda y}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{m} x \frac{\partial}{\partial y} \right) f.$$

The adjoint equation on the phase space is:

$$\dot{f} = \left(\left(mk^2 q + \frac{\lambda}{2} q^2 \right) \frac{\partial}{\partial p} - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The last equation is the classical Hamilton equation generated by the cubic potential (7.27). Qualitative analysis of its dynamics can be found in many textbooks [4, § 4.C, Pic. 12; 109, § 4.4].

Remark 7.23. We have obtained the *Poisson bracket* from the commutator of convolutions on \mathbb{H}^n without any quasiclassical limit $\hbar \rightarrow 0$. This has a common source with the deduction of main calculus theorems in [17] based on dual numbers. As explained in [82, Rem. 6.9] this is due to the similarity between the parabolic unit ε and the infinitesimal number used in non-standard analysis [23]. In other words, we never need to take care about terms of order $O(\hbar^2)$ because they will be wiped out by $\varepsilon^2 = 0$.

An alternative derivation of classical dynamics from the Heisenberg group is given in the recent paper [100].

7.6.3. Classical probabilities. It is worth to notice that dual numbers are not only helpful in reproducing classical Hamiltonian dynamics, they also provide the classic rule for addition of probabilities. We use the same formula (7.29) to calculate kernels of the states. The important difference now that the representation (7.65) does not change the support of functions. Thus if we calculate the correlation term $\langle v_1, \rho(g)v_2 \rangle$ in (7.30), then it will be zero for every two vectors v_1 and v_2 which have disjoint supports in the phase space. Thus no interference similar to quantum or hyperbolic cases (Subsection 7.3.3) is possible.

7.6.4. Ladder Operator for the Nilpotent Subgroup. Finally we look for ladder operators for the Hamiltonian $\tilde{B} + \tilde{Z}/2$ or, equivalently, $-\tilde{B} + \tilde{Z}/2$. It can be identified with a free particle [124, § 3.8].

We can look for ladder operators in the representation (7.12–7.13) within the Lie algebra \mathfrak{h}_1 in the form $L_\varepsilon^\pm = a\tilde{X} + b\tilde{Y}$. This is possible if and only if

$$(7.68) \quad -b = \lambda a, \quad 0 = \lambda b.$$

The compatibility condition $\lambda^2 = 0$ implies $\lambda = 0$ within complex numbers. However such a “ladder” operator produces only the zero shift on the eigenvectors, cf. (3.15).

Another possibility appears if we consider the representation of the Heisenberg group induced by dual-valued characters. On the configuration space such a representation is [87, (4.11)]:

$$(7.69) \quad [\rho_\chi^\varepsilon(s, x, y)f](q) = e^{2\pi i x q} \left(\left(1 - \varepsilon h(s - \tfrac{1}{2}xy)\right) f(q) + \frac{\varepsilon h y}{2\pi i} f'(q) \right).$$

The corresponding derived representation of \mathfrak{h}_1 is

$$(7.70) \quad \rho_h^p(X) = 2\pi i q, \quad \rho_h^p(Y) = \frac{\varepsilon h}{2\pi i} \frac{d}{dq}, \quad \rho_h^p(S) = -\varepsilon h I.$$

However the Shale–Weil extension generated by this representation is inconvenient. It is better to consider the FSB-type parabolic representation (7.65) on the phase space induced by the same dual-valued character. Then the derived representation of \mathfrak{h}_1 is:

$$(7.71) \quad \rho_h^p(X) = -2\pi i q - \frac{\varepsilon h}{4\pi i} \partial_p, \quad \rho_h^p(Y) = -2\pi i p + \frac{\varepsilon h}{4\pi i} \partial_q, \quad \rho_h^p(S) = \varepsilon h I.$$

An advantage of the FSB representation is that the derived form of the parabolic Shale–Weil representation coincides with the elliptic one (7.19).

Eigenfunctions with the eigenvalue μ of the parabolic Hamiltonian $\tilde{B} + \tilde{Z}/2 = q\partial_p$ have the form

$$(7.72) \quad v_\mu(q, p) = e^{\mu p/q} f(q), \text{ with an arbitrary function } f(q).$$

The linear equations defining the corresponding ladder operator $L_\varepsilon^\pm = a\tilde{X} + b\tilde{Y}$ in the algebra \mathfrak{h}_1 are (7.68). The compatibility condition $\lambda^2 = 0$ implies $\lambda = 0$ within complex numbers again. Admitting dual numbers we have additional values $\lambda = \pm\varepsilon\lambda_1$ with $\lambda_1 \in \mathbb{C}$ with the corresponding ladder operators

$$L_\varepsilon^\pm = \tilde{X} \mp \varepsilon\lambda_1\tilde{Y} = -2\pi i q - \frac{\varepsilon h}{4\pi i} \partial_p \pm 2\pi\varepsilon\lambda_1 i p = -2\pi i q + \varepsilon i(\pm 2\pi\lambda_1 p + \frac{h}{4\pi} \partial_p).$$

For the eigenvalue $\mu = \mu_0 + \varepsilon\mu_1$ with $\mu_0, \mu_1 \in \mathbb{C}$ the eigenfunction (7.72) can be rewritten as:

$$(7.73) \quad v_\mu(q, p) = e^{\mu p/q} f(q) = e^{\mu_0 p/q} \left(1 + \varepsilon\mu_1 \frac{p}{q} \right) f(q)$$

due to the nilpotency of ε . Then the ladder action of L_ε^\pm is $\mu_0 + \varepsilon\mu_1 \mapsto \mu_0 + \varepsilon(\mu_1 \pm \lambda_1)$. Therefore these operators are suitable for building \mathfrak{sp}_2 -modules with a one-dimensional chain of eigenvalues.

Finally, consider the ladder operator for the same element $B + Z/2$ within the Lie algebra \mathfrak{sp}_2 , cf. § 3.3.3. There is the only operator $L_p^\pm = \tilde{B} + \tilde{Z}/2$ corresponding to complex coefficients, which does not affect the eigenvalues. However the dual numbers lead to the operators

$$L_\varepsilon^\pm = \pm\varepsilon\lambda_2\tilde{A} + \tilde{B} + \tilde{Z}/2 = \pm\frac{\varepsilon\lambda_2}{2} (q\partial_q - p\partial_p) + q\partial_p, \quad \lambda_2 \in \mathbb{C}.$$

These operator act on eigenvalues in a non-trivial way.

7.6.5. Similarity and Correspondence. We wish to summarise our findings. Firstly, the appearance of hypercomplex numbers in ladder operators for \mathfrak{h}_1 follows exactly the same pattern as was already noted for \mathfrak{sp}_2 , see Rem. 3.8:

- the introduction of complex numbers is a necessity for the *existence* of ladder operators in the elliptic case;
- in the parabolic case we need dual numbers to make ladder operators *useful*;
- in the hyperbolic case double numbers are not required neither for the existence or for the usability of ladder operators, but they do provide an enhancement.

In the spirit of the Similarity and Correspondence Principle 3.5 we have the following extension of Prop. 3.9:

Proposition 7.24. *Let a vector $H \in \mathfrak{sp}_2$ generates the subgroup K , N' or A' , that is $H = Z$, $B + Z/2$, or $2B$ respectively. Let ι be the respective hypercomplex unit. Then the ladder operators L^\pm satisfying to the commutation relation:*

$$[H, L_2^\pm] = \pm\iota L^\pm$$

are given by:

- (i) Within the Lie algebra \mathfrak{h}_1 : $L^\pm = \tilde{X} \mp \iota\tilde{Y}$.
- (ii) Within the Lie algebra \mathfrak{sp}_2 : $L_2^\pm = \pm\iota\tilde{A} + \tilde{E}$. Here $E \in \mathfrak{sp}_2$ is a linear combination of B and Z with the properties:
 - $E = [A, H]$.
 - $H = [A, E]$.
 - Killings form $K(H, E)$ [55, § 6.2] vanishes.
 Any of the above properties defines the vector $E \in \text{span}\{B, Z\}$ up to a real constant factor.

It is worth continuing this investigation and describing in details hyperbolic and parabolic versions of FSB spaces.

8. OPEN PROBLEMS

A reader may already note numerous objects and results, which deserve a further consideration. It may also worth to state some open problems explicitly. In this section we indicate several directions for further work, which go through four main areas described in the paper.

8.1. Geometry. Geometry is most elaborated area so far, yet many directions are waiting for further exploration.

- (i) Möbius transformations (1.1) with three types of hypercomplex units appear from the action of the group $SL_2(\mathbb{R})$ on the homogeneous space $SL_2(\mathbb{R})/H$ [85], where H is any subgroup A, N, K from the Iwasawa decomposition (1.3). Which other actions and hypercomplex numbers can be obtained from other Lie groups and their subgroups?
- (ii) Lobachevsky geometry of the upper half-plane is extremely beautiful and well-developed subject [8, 22]. However the traditional study is limited to one subtype out of nine possible: with the complex numbers for Möbius transformation and the complex imaginary unit used in FSCc (2.8). The remaining eight cases shall be explored in various directions, notably in the context of discrete subgroups [7].
- (iii) The Fillmore-Springer-Cnops construction, see subsection 2.2, is closely related to the *orbit method* [57] applied to $SL_2(\mathbb{R})$. An extension of the orbit method from the Lie algebra dual to matrices representing cycles may be fruitful for semisimple Lie groups.
- (iv) A development of a discrete version of the geometrical notions can be derived from suitable discrete groups. A natural first example is the group $SL_2(\mathbb{F})$, where \mathbb{F} is a finite field, e.g. \mathbb{Z}_p the field of integers modulo a prime p .

8.2. Analytic Functions. It is known that in several dimensions there are different notions of analyticity, e.g. several complex variables and Clifford analysis. However, analytic functions of a complex variable are usually thought to be the only options in a plane domain. The following seems to be promising:

- (i) Development of the basic components of analytic function theory (the Cauchy integral, the Taylor expansion, the Cauchy-Riemann and Laplace equations, etc.) from the same construction and principles in the elliptic, parabolic and hyperbolic cases and respective subcases.
- (ii) Identification of Hilbert spaces of analytic functions of Hardy and Bergman types, investigation of their properties. Consideration of the corresponding Toeplitz operators and algebras generated by them.
- (iii) Application of analytic methods to elliptic, parabolic and hyperbolic equations and corresponding boundary and initial values problems.
- (iv) Generalisation of the results obtained to higher dimensional spaces. Detailed investigation of physically significant cases of three and four dimensions.
- (v) There is a current interest in construction of analytic function theory on discrete sets. Our approach is ready for application to an analytic functions in discrete geometric set-up outlined in item 8.1.iv above.

8.3. Functional Calculus. The functional calculus of a finite dimensional operator considered in Section 6 is elementary but provides a coherent and comprehensive treatment. It shall be extended to further cases where other approaches seems to be rather limited.

- (i) Nilpotent and quasinilpotent operators have the most trivial spectrum possible (the single point $\{0\}$) while their structure can be highly non-trivial. Thus the standard spectrum is insufficient for this class of operators. In contract, the covariant calculus and the spectrum give complete description of nilpotent operators—the basic prototypes of quasinilpotent ones. For quasinilpotent operators the construction will be more complicated and shall use analytic functions mentioned in 8.2.i.
- (ii) The version of covariant calculus described above is based on the *discrete series* representations of $SL_2(\mathbb{R})$ group and is particularly suitable for the description of the *discrete spectrum* (note the remarkable coincidence in the names).

It is interesting to develop similar covariant calculi based on the two other representation series of $SL_2(\mathbb{R})$: *principal* and *complementary* [96]. The corresponding versions of analytic function theories for principal [64] and complementary series [82] were initiated within a unifying framework. The classification of analytic function theories into elliptic, parabolic, hyperbolic [78, 82] hints the following associative chains:

Representations	— Function Theory —	Type of Spectrum
discrete series	— elliptic —	discrete spectrum
principal series	— hyperbolic —	continuous spectrum
complementary series	— parabolic —	residual spectrum

- (iii) Let a be an operator with $\text{sp } a \in \mathbb{D}$ and $\|a^k\| < Ck^p$. It is typical to consider instead of a the *power bounded* operator ra , where $0 < r < 1$, and consequently develop its H_∞ calculus. However such a regularisation is very rough and hides the nature of extreme points of $\text{sp } a$. To restore full information a subsequent limit transition $r \rightarrow 1$ of the regularisation parameter r is required. This make the entire technique rather cumbersome and many results have an indirect nature.

The regularisation $a^k \rightarrow a^k/k^p$ is more natural and accurate for polynomially bounded operators. However it cannot be achieved within the homomorphic calculus Defn. 6.1 because it is not compatible with any algebra homomorphism. Albeit this may be achieved within the covariant calculus Defn. 6.4 and Bergman type space from 8.2.ii.

- (iv) Several non-commuting operators are especially difficult to treat with functional calculus Defn. 6.1 or a joint spectrum. For example, deep insights on joint spectrum of commuting tuples [116] refused to be generalised to non-commuting case so far. The covariant calculus was initiated [62] as a new approach to this hard problem and was later found useful elsewhere as well. Multidimensional covariant calculus [73] shall use analytic functions described in 8.2.iv.
- (v) As we noted above there is a duality between the co- and contravariant calculi from Defns. 4.20 and 4.22. We also seen in Section 6 that functional calculus is an example of contravariant calculus and the functional model is a case of a covariant one. It is interesting to explore the duality between them further.

8.4. Quantum Mechanics. Due to the space restrictions we only touched quantum mechanics, further details can be found in [63, 74, 76, 77, 79, 87]. In general, Erlangen approach is much more popular among physicists rather than mathematicians. Nevertheless its potential is not exhausted even there.

- (i) There is a possibility to build representation of the Heisenberg group using characters of its centre with values in dual and double numbers rather

- than in complex ones. This will naturally unifies classical mechanics, traditional QM and hyperbolic QM [52]. In particular, a full construction of the corresponding Fock–Segal–Bargmann spaces would be of interest.
- (ii) Representations of nilpotent Lie groups with multidimensional centres in Clifford algebras as a framework for consistent quantum field theories based on De Donder–Weyl formalism [76].

Remark 8.1. This work is performed within the “Erlangen programme at large” framework [78, 82], thus it would be suitable to explain the numbering of various papers. Since the logical order may be different from chronological one the following numbering scheme is used:

Prefix	Branch description
“0” or no prefix	Mainly geometrical works, within the classical field of Erlangen programme by F. Klein, see [82, 85]
“1”	Papers on analytical functions theories and wavelets, e.g. [64]
“2”	Papers on operator theory, functional calculi and spectra, e.g. [75]
“3”	Papers on mathematical physics, e.g. [87]

For example, [87] is the first paper in the mathematical physics area. The present paper [89] outlines the whole framework and thus does not carry a subdivision number. The on-line version of this paper may be updated in due course to reflect the achieved progress.

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INDEX

- A subgroup, 3
- A-orbit, 3
- K subgroup, 3
- K-orbit, 3
- N subgroup, 3
- N-orbit, 3
- $SL_2(\mathbb{R})$ group, 2
 - representation, 14–16
 - in Banach space, 35
- $Sp(2)$, 45
- \mathbb{H}^1 , Heisenberg group, 44
- \mathbb{J}^n (jet space), 36
- ι (hypercomplex unit), 3
- \check{i} (hypercomplex unit in cycle space), 6
- j (hyperbolic unit), 3
- i (imaginary unit), 3
- ε (parabolic unit), 3
- σ ($\sigma := \iota^2$), 3
- $\check{\sigma}$ ($\check{\sigma} := \check{i}^2$), 6
- $ax + b$ group, 2, 20, 22, 25–27
 - invariant measure, 21
 - representation, 21
- k-normalised cycle, 11
- action
 - derived, 3
 - transitive, 2
- admissible wavelet, 19, 20, 25
- affine group, *see also* $ax + b$ group
- algebra
 - Clifford, 12, 22, 44, 63, 65
 - homomorphism, 33
- analysis
 - multiresolution, 22
- analytic
 - contravariant calculus, 34
 - function on discrete sets, 63
- annihilation operator, *see also* ladder operator
- annulus, 32
- Atiyah–Singer index theorem, 3
- Bergman
 - space, 4, 19, 21, 33, 63
- Blaschke
 - product, 38
- boundary effect on the upper half-plane, 8
- bracket
 - Moyal, 50, 51
 - hyperbolic, 42, 55
 - Poisson, 61
- calculus
 - contravariant, 24, 64
 - analytic, 34
 - covariant, 24, 38, 40, 41, 64
 - Berezin, 23
 - functional, 4, 24, 33, 63, 64
 - Riesz–Dunford, 24
 - support, *see also* spectrum
 - symbolic, *see also* covariant calculus
 - umbral, 20
- cancellative semigroup, 20
- Cartan
 - subalgebra, 17
- case
 - elliptic, 3
 - hyperbolic, 3
 - parabolic, 3
- Casimir operator, 30
- categorical viewpoint, 8
- category theory, 8
- Cauchy
 - integral, 4, 21, 30–32, 35, 63
- Cauchy–Riemann operator, 4, 27, 29, 30, 53, 63
- causal, 48
- Cayley transform, 31
- centre
 - length from, 12
 - of a cycle, 7
- characteristic
 - function, 23
- classic
 - Fock–Segal–Bargmann representation, 59
 - probability, 61
 - Schrödinger representation, 59
- classical mechanics, 44, 49
- Clifford
 - algebra, 12, 22, 44, 63, 65
- Clifford algebras, 12
- coherent state, *see also* wavelet
- commutator, 42, 47
- commutator relation, 44
- concentric, 7
- cone
 - generator, 6
- configuration
 - space, 49, 51, 55, 59, 61
- conformality, 12
- conic
 - section, 3, 6
- conjugation
 - cycles, of, 10
- constant
 - Planck, 42, 60
- construction
 - Fillmore–Springer–Cnops, 7, 63
 - operator, 41
 - Gelfand–Naimark–Segal, 9
- contravariant
 - calculus, 24, 64
 - analytic, 34
 - spectrum, 33, 34, 36, 38
 - stability, 38
 - symbol, 24
- convolution, 47
- correspondence, *see also* principle of similarity
 - and correspondence
- covariant
 - calculus, 24, 38, 40, 41, 64
 - Berezin, 23
 - pencil, 40
 - spectral distance, 39

- symbol, 23
- symbolic calculus, 23
- transform, 19
 - induced, 26
- covariant transform
 - inverse, 24
- creation operator, *see also* ladder operator
- cycle, 5, 42
 - centre, 7
 - conjugation, 10
 - equation, 6
 - focus, 7
 - isotropic, 10
 - matrix, 6
 - normalised, 8
 - k-, 11
 - Kirillov, 8
 - radius, 11
 - reflection, 10
 - space, 6
 - zero-radius, 8, 10
- cycles
 - f-orthogonal, 10, 12
 - focal orthogonal, *see also* f-orthogonal
 - orthogonal, 7, 9
- De Donder–Weyl formalism, 65
- decomposition
 - Iwasawa, 3, 63
- defect operator, 23
- derivation
 - inner, 42, 47
- derived action, 3
- determinant, 8
- directing functional, 22
- directrix, 7
- discrete
 - analytic function, 63
 - geometry, 63
 - spectrum, 64
- distance, 11
 - covariant spectral, 39
- divisor
 - zero, 3, 58
- domain
 - non-simply connected, 32
 - simply connected, 32
- double, 3
 - number, 3, 13, 17, 18, 42, 54, 58
- drum
 - hearing shape, 8
- dual, 3
 - number, 3, 13–16, 18, 42, 58, 58–62
- eigenvalue, 17, 36, 38
 - generalised, 39
 - quadratic, 41
- elliptic
 - case, 3
- EPH-classification, 3
- equation
 - Cauchy–Riemann, *see also* Cauchy–Riemann operator
 - dynamics in p-mechanics, 47
 - Hamilton, 42, 60
 - Heisenberg, 42, 50
 - Erlangen programme, 2, 2
- f-orthogonality, 10, 12
- factorisation
 - Wiener–Hopf, 22
- fiducial operator, 19
- field
 - quantum, 65
- Fillmore–Springer–Cnops construction, 7, 63
 - operator, 41
- focal
 - orthogonality, *see also* f-orthogonality
- Fock–Segal–Bargmann
 - representation, 45, 46, 61
 - classic (parabolic), 59
 - hyperbolic, 55
 - representations, 49
 - space, 22, 46, 53, 62, 65
- focus
 - length from, 12
 - of a cycle, 7
 - of a parabola, 7
- form
 - symplectic, 44
- formula
 - reconstruction, 25
- FSB
 - representation, *see also*
 - Fock–Segal–Bargmann representation
 - space, *see also* Fock–Segal–Bargmann space
- function
 - characteristic, 23
 - Heaviside, 10
 - Hermite, *see also* Hermite polynomial
 - maximal, 22
 - parabolic cylinder, *see also* Weber–Hermite function
 - polyanalytic, 30
 - Weber–Hermite, 56, 58
- functional
 - calculus, 4, 24, 33, 63, 64
 - Riesz–Dunford, 24
 - support, *see also* spectrum
 - directing, 22
 - invariant, 24
 - model, 23, 33, 38, 64
- Gaussian, 21, 51, 54, 56
- Gelfand–Naimark–Segal construction, 9
- generalised
 - eigenvalue, 39
- generator
 - of a cone, 6
 - of a subgroup, 17
 - quadratic, 46
- geometrical quantisation, 42
- geometry, 63
 - discrete, 63
 - Lobachevsky, 3, 25, 29, 63
 - non-commutative, 4, 42

- GNS construction, 9, *see also*
 - Gelfand–Naimark–Segal construction
- group, 2
 - $SL_2(\mathbb{R})$, 2
 - $Sp(2)$, 43, *see also also* $SL_2(\mathbb{R})$
 - $SU(1, 1)$, 23, *see also also* $SL_2(\mathbb{R})$
 - $\alpha x + b$, 2, 20, 22, 25–27
 - invariant measure, 21
 - affine, *see also* $\alpha x + b$ group
 - Heisenberg, 2, 42, 44–62, 64
 - Fock–Segal–Bargmann representation, 45
 - induced representation, 44
 - invariant measure, 44
 - representation
 - linear, 4
 - Schrödinger, 45, 52
 - symplectic, 44, 53
- Haar measure, *see also* invariant measure
- half-plane, 3, 40
 - boundary effect, 8
- Hamilton
 - equation, 42, 60
- Hardy
 - maximal function, 22
 - pairing, 25
 - space, 4, 19, 21, 30, 32, 34, 39, 63
- harmonic
 - oscillator, 44, 46, 47, 50, 52, 55, 60
 - repulsive (hyperbolic), 56
- hearing drum’s shape, 8
- Heaviside
 - function, 10
- Heaviside function, 10
- Heisenberg
 - commutator relation, 44
 - equation, 42, 50
 - group, 2, 42, 44–62, 64
 - Fock–Segal–Bargmann representation, 45
 - induced representation, 44
 - invariant measure, 44
- Heisenberg group, 2
- Hermite
 - polynomial, 53, 58
- homomorphism
 - algebraic, 33
- hyperbolic
 - case, 3
 - Fock–Segal–Bargmann representation, 55
 - harmonic oscillator, 56
 - Moyal bracket, 42, 55
 - probability, 56
 - Schrödinger representation, 55
 - unharmonic oscillator, 55
 - unit (j), 3
- hypercomplex
 - number, 3, 16, 18, 62
- imaginary
 - unit (i), 3
- induced
 - covariant transform, 26
 - representation, 13, 35, 42
 - Heisenberg group, of, 44
- induction, 13
- inner
 - derivation, 42, 47
- integral
 - Cauchy, 4, 21, 30–32, 35, 63
 - Poisson, *see also* Poisson kernel
- interference, 48, 51, 61
- intertwining operator, 4, 20, 22, 25, 26, 34, 40
- invariant, 24
 - functional, 24
 - joint, 9, 10
 - measure, 19–21, 24, 25, 44
 - pairing, 24
- inverse
 - covariant transform, 24
- isotropic cycles, 10
- Iwasawa decomposition, 3, 63
- jet, 4, 36, 37, 39, 42
 - bundle, 36
- joint
 - invariant, 9, 10
 - spectrum, 34
- Jordan
 - normal form, 36
- kernel
 - Poisson, 21, 30
- Krein
 - directing functional, 22
 - space, 16, 42
- ladder operator, 17–19, 29, 31, 53–54, 56–58, 61–62
- Laplace operator, *see also* Laplacian
- Laplacian, 4, 29, 30, 63
- left regular representation, 44
- length, 12
 - from centre, 12
 - from focus, 12
- Lidskii theorem, 38
- limit
 - semiclassical, 42, 49
- Littlewood–Paley
 - operator, 22
- Lobachevsky
 - geometry, 3, 25, 29, 63
- lowering operator, *see also* ladder operator
- Möbius map, 2, 21, 23, 28, 32, 38, 40, 42
 - on cycles, 7
- map
 - Möbius, 2, 21, 23, 28, 32, 38, 40, 42
 - on cycles, 7
- matrix
 - cycle, of a, 6
 - Jordan normal form, 36
- maximal
 - function, 22
- measure
 - Haar, *see also* invariant measure
 - invariant, 19–21, 24, 25, 44
- mechanics

- classical, 44, 49
- metaplectic representation, *see also* oscillator representation
- method
 - orbits, of, 42, 63
- minimal
 - polynomial, 38
- model
 - functional, 23, 33, 38, 64
- mother wavelet, 20, 35
- Moyal bracket, 50, 51
 - hyperbolic, 42, 55
- multiresolution analysis, 22
- nilpotent unit, *see also* parabolic unit
- non-commutative geometry, 4, 42
- non-simply connected domain, 32
- norm
 - shift invariant, 22
- normalised
 - cycle, 8
 - k-, 11
 - Kirillov, 8
- number
 - double, 3, 13, 17, 18, 42, 54, 58
 - dual, 3, 13–16, 18, 42, 58, 58–62
 - hypercomplex, 3, 16, 18, 62
- numerical
 - range, 23
- observable, 47
- on hearing drum's shape, 8
- operator
 - annihilation, *see also* ladder operator
 - Casimir, 30
 - Cauchy-Riemann, 4, 27, 29, 30, 53, 63
 - creation, *see also* ladder operator
 - defect, 23
 - fiducial, 19
 - Fillmore–Springer–Cnops construction, 41
 - intertwining, 4, 20, 22, 25, 26, 34, 40
 - ladder, 17–19, 29, 31, 53–54, 56–58, 61–62
 - Laplace, *see also* Laplacian
 - Littlewood–Paley, 22
 - lowering, *see also* ladder operator
 - power bounded, 64
 - pseudodifferential, 24
 - quasinilpotent, 64
 - raising, *see also* ladder operator
 - Toeplitz, 63
- optics, 45
- orbit
 - method, 42, 63
 - subgroup A, of, 3
 - subgroup K, of, 3
 - subgroup N, of, 3
- orders
 - of zero, 38
- orthogonality
 - cycles, of, 7, 9
 - focal, *see also* f-orthogonality
- oscillator
 - harmonic, 44, 46, 47, 50, 52, 55, 60
 - repulsive (hyperbolic), 56
 - representation, 45
 - unharmonic, 48, 50, 60
 - hyperbolic, 55
- p-mechanics, 46
 - dynamic equation, 47
 - observable, 47
 - state, 48
- p-mechanisation, 47
- pairing
 - Hardy, 25
 - invariant, 24
- parabola
 - directrix, 7
 - focus, 7
 - vertex, 7
- parabolic
 - case, 3
 - cylinder function, *see also* Weber–Hermite function
 - Fock–Segal–Bargmann representation, 59
 - probability, *see also* classic probability
 - Schrödinger representation, 59
 - unit (ε), 3
- PDO, 24
- pencil
 - covariant, 40
- perpendicular, 12
- phase
 - space, 14, 43, 46, 49, 51, 52, 55
- Planck
 - constant, 42, 60
- Plato's cave, 7
- point space, 6
- Poisson
 - bracket, 61
 - kernel, 21, 30
- polyanalytic function, 30
- polynomial
 - Hermite, 53, 58
 - minimal, 38
- power bounded operator, 64
- primary
 - representation, 34, 36, 41
- principle
 - similarity and correspondence, 16, 18, 57, 62
- probability
 - classic (parabolic), 61
 - hyperbolic, 56
 - quantum, 43, 48, 51
- product
 - Blaschke, 38
- projective space, 6
- prolongation, 35
- pseudodifferential operator, 24
- \mathcal{PT} -symmetry, 42
- quadratic
 - eigenvalue, 41
 - generator, 46
- quantisation

- geometrical, 42
- quantum
 - field, 65
 - probability, 43, 48, 51
- quantum mechanics, 44
- quasinilpotent
 - operator, 64
- quaternion, 12
- quaternions, 12
- radius
 - cycle, of, 11
- Radon transform, 22
- raising operator, *see also* ladder operator
- range
 - numerical, 23
- reconstruction formula, 25
- reduced wavelet transform, 26
- reflection
 - in a cycle, 10
- representation
 - $SL_2(\mathbb{R})$ group, 14–16
 - in Banach space, 35
 - $\alpha x + b$ group, 21
 - coefficients, *see also* wavelet transform
 - complementary series, 4, 21, 64
 - discrete series, 4, 15, 19, 21, 64
 - Fock–Segal–Bargmann, 45, 46, 61
 - classic (parabolic), 59
 - hyperbolic, 55
 - FSB, *see also* Fock–Segal–Bargmann
 - representation
 - Heisenberg group
 - classic (parabolic), 59
 - hyperbolic, 55
 - Schrödinger, 45
 - induced, 13, 35, 42
 - Heisenberg group, of, 44
 - left regular, 44
 - linear, 4
 - metaplectic, *see also* oscillator
 - representation
 - oscillator, 45
 - primary, 34, 36, 41
 - principal series, 4, 21, 64
 - Schrödinger, 45, 49, 49, 53, 57
 - classic (parabolic), 59
 - hyperbolic, 55
 - Shale–Weil, *see also* oscillator representation
 - square integrable, 19, 20, 24–26
- representations
 - Fock–Segal–Bargmann, 49
 - linear, 13
- repulsive
 - harmonic oscillator, 56
- resolvent, 24, 33, 35, 41
- Riemann
 - mapping theorem, 32
- Schrödinger
 - group, 45, 52
 - representation, 45, 49, 49, 53, 57
 - classic (parabolic), 59
 - hyperbolic, 55
- sections
 - conic, 3, 6
- semiclassical
 - limit, 42, 49
- semigroup
 - cancellative, 20
- Shale–Weil representation, *see also* oscillator representation
- similarity, *see also* principle of similarity and correspondence
- simply connected domain, 32
- space
 - Bergman, 4, 19, 21, 33, 63
 - configuration, 49, 51, 55, 59, 61
 - cycles, of, 6
 - Fock–Segal–Bargmann, 22, 46, 53, 62, 65
 - FSB, *see also* Fock–Segal–Bargmann space
 - Hardy, 4, 19, 21, 30, 32, 34, 39, 63
 - Krein, 16, 42
 - phase, 14, 43, 46, 49, 51, 52, 55
 - point, 6
 - projective, 6
- spectral
 - covariant distance, 39
- spectrum, 33, 35
 - contravariant, 33, 34, 36, 38
 - stability, 38
 - discrete, 64
 - joint, 34
 - mapping, 33
- square integrable
 - representation, 19, 20, 24–26
- stability
 - contravariant spectrum, 38
- state, 48
 - coherent, *see also* wavelet
 - vacuum, *see also* mother wavelet
- subalgebra
 - Cartan, 17
- subgroup
 - A, 3, 5
 - orbit, 3
 - K, 3, 5
 - orbit, 3
 - N, 3, 5
 - orbit, 3
 - generator, 17
- supersymmetry, 16
- support
 - functional calculus, *see also* spectrum
- symbol
 - contravariant, 24
 - covariant, 23
- symbolic calculus, *see also* covariant calculus
 - covariant, 23
- symplectic
 - form, 44
 - group, 44, 53
 - transformation, 13, 14, 43, 56
- theorem
 - Atiyah–Singer (index), 3

- Lidskii, 38
- Riemann mapping, 32
- spectral mapping, 33
- Toeplitz
 - operator, 63
- token, 20
- trace, 8
- transform
 - Cayley, 31
 - covariant, 19
 - induced, 26
 - inverse, 24
 - Radon, 22
 - wavelet, *see also* wavelet transform
- transformation
 - symplectic, 13, 14, 43, 56
- transitive, 2
- umbral calculus, 20
- unharmonic
 - oscillator, 48, 50, 60
 - hyperbolic, 55
- unit
 - circle, 31
 - disk, 31
 - hyperbolic (j), 3
 - imaginary (i), 3
 - nilpotent, *see also* parabolic unit
 - parabolic (ϵ), 3
- vacuum state, *see also* mother wavelet
- vertex
 - of a parabola, 7
- wavelet, 19, 20, 25, 28, 35
 - admissible, 19, 20, 25
 - mother, 20, 35
 - transform, 20, 35, 53
 - induced, *see also* induced covariant transform
 - reduced, 26
- Weber–Hermite function, 56, 58
- Wiener–Hopf factorisation, 22
- zero
 - divisor, 3, 58
 - order, of, 38
- zero-radius cycle, 8, 10

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